ON CERTAIN SUBMANIFOLDS OF AN H-STRUCTURE MANIFOLD

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Abstract

Submanifolds of codimension 2 of an almost hyperbolic Hermite manifold have been studied by Rai[4] and others. In this paper, we have taken an H-structure manifold and showed that its submanifold of codimension r admits the generalised para (ε,r) contact structure. Certain other useful results have also been proved in this paper.

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1. Preliminaries

Let M^n be an n-dimensional differential manifold of class C^∞. Suppose there exists on M^n a tensor field $F(\neq 0)$ of type (1,1) satisfying

$$F^2 = a^2 I \quad (1.1)$$

where'a' is a non-zero complex number. Suppose further that above M^n also admits a Hermite metric G such that

$$G(FX^*, FY^*) + a^2 G(X^*, Y^*) = 0 \quad (1.2)$$

holds for arbitrary vector fields X^* and Y^* on M^n. Then the manifold M^n satisfying (1.1) and (1.2) will be called an H-structure manifold.
Let $'F(X^*, Y^*)$ be the tensor field of type (0,2) given by

$$'F(X^*, Y^*) = G(FX^*, Y^*) \quad (1.3)$$

The following results can be proved easily

$$\begin{aligned}&(i)'F(FX^*, Y^*) = -'f(X^*, FY^*) = a^2 G(X^*, Y^*)\\&(ii)'F(X^*, Y^*) + 'F(Y^*, X^*) = 0\\&\text{and}\\&(iii)'F(FX^*, FY^*) + a^2{}'F(X^*, Y^*) = 0\end{aligned} \quad (1.4)$$

Let $\widetilde{D}$ be the Riemannian connection of M^n.
Thus,

$$\widetilde{D}_{X^*}Y^* - \widetilde{D}_{Y^*}X^* = [X^*, Y^*] \quad (1.5)$$

and

$$\tilde{D}_{X^*}G = 0 \quad (1.6)$$

If $\tilde{N}(X^*, Y^*)$ be the Nijenhuis tensor formed with F, we have

$$\tilde{N}(X^*, Y^*) = [FX^*, FY^*] - F[FX^*, Y^*] - F[X^*, FY^*] + F^2[X^*, Y^*] \quad (1.7)$$

An H-structure manifold M^n will be called a K-manifold if the structure tensor F is parallel i.e.

$$(D_{X^*}F)(Y^*) = 0 \quad (1.8)$$

A submanifold M^{n-r} of codimension r immersed in the H-structure manifold M^n will be said to possess a generalised para (ε, r)-contact structure if there exists on M^{n-r} a tensor field f of type (1,1), $r(C^\infty)$ contravariant vector field $\underset{x}{U}$ $r(C^\infty)$ I-forms u^x (r some finite integer) and a constant ε such that

$$f^2 = a^2 I - \sqrt{\varepsilon}\sum_{x=1}^{r} \overset{x}{u} \otimes \underset{x}{U} \quad (1.9)$$

Also

$$\begin{aligned} &(i) f\underset{x}{U} + \sqrt{\varepsilon}\sum_{y=1}^{r} \theta_x^y \underset{y}{U} = 0 \\ &(ii) \overset{y}{u} o f + \sqrt{\varepsilon}\sum_{x=1}^{r} \theta_x^y \overset{x}{u} = 0 \\ &(iii) u^z\left(\underset{x}{U}\right) + \sqrt{\varepsilon}\sum_{y=1}^{r} \theta_y^z \theta_x^y = \frac{a^2}{\sqrt{\varepsilon}}\delta_x^z \end{aligned} \quad (1.10)$$

where x,y,z=1,2,...,r, δ_y^x denotes the Kronecker delta and θ_x^y are scalar fields.
If in addition, the above M^{n-r} also admits a metric tensor 'g' satisfying

$$g(fX, fY) + a^2 g(X, Y) + \varepsilon\sum_{x=1}^{r} \overset{x}{u}(X)\overset{x}{u}(Y) = 0 \quad (1.11)$$

We say that the manifold M^{n-r} admits a generalised para (ε, r)-contact metric structure.
A vector field V^* on M^n will be called a contravariant almost analytic vector field if .

$$(L_{V^*}F)(X^*) = 0 \quad (1.12)$$

where L denotes the Lie-differentiation. For a Kaehler manifold, the almost analytic vector field satisfies.

$$FD_{X^*}V^* + D_X^* V^* = 0 \quad (1.13)$$

2. Submanifolds of Codimension r

Let M^{n-r} be a submanifold of codimension r immersed differentiably in H-structure manifold M^n. If b denotes the differential of the immersion, a vector field X in the tangent space of M^{n-r} corresponds to a vector field BX in that of M^n . If $N_x, x = 1,2,\ldots,r$ denotes the mutually orthogonal set of unit normals to M^{n-r} and 'g' the induced metric on M^{n-r} we can write

$$\begin{aligned} &(i) G(BX, BY) = g(X, Y), \\ &(ii) G\left(BX, \underset{x}{N}\right) = 0 \\ &\text{and} \\ &(iii) G\left(\underset{x}{N}, \underset{y}{N}\right) = \delta_{xy} \end{aligned} \quad (2.1)$$

where x,y=1,2,...,r and δ_{xy} denotes the Kronecker delta.
We can write the transformation for FBX and $F\underset{x}{N}$ as [3]

$$FBX = BfX - \sqrt{\varepsilon}\sum_{x=1}^{r} \overset{x}{u}(X)\underset{x}{N} \tag{2.2}$$

and

$$F\underset{x}{N} = -B\underset{x}{U} + \sqrt{\varepsilon}\sum_{y=1}^{r} \theta_x^y \underset{y}{N} \tag{2.3}$$

where f is a tensor field of type (1,1), $\overset{x}{u}$ are 1-forms and $\underset{x}{U}$ vector fields on the submanifolds M^{n-r}, x=1,2,...,r.
Operating (2.2) by F and making use of the equations (1.1), (2.2) and (2.3), we obtain

$$a^2BX = Bf^2X - \sqrt{\varepsilon}\sum_{y=1}^{r} \overset{y}{u}(fX)\underset{y}{N} - \sqrt{\varepsilon}\sum_{x=1}^{r} \overset{x}{u}(X) - B\underset{x}{U} + \sqrt{\varepsilon}\sum_{y=1}^{r} \theta_x^y \underset{y}{N}$$

Comparison of tangential and normal vectors yields

$$(i) f^2 = a^2I - \sqrt{\varepsilon}\sum_{x=1}^{r} \overset{x}{u}\otimes\underset{x}{U}$$
and
$$(ii) \overset{y}{u}oF + \sqrt{\varepsilon}\sum_{x=1}^{r} \theta_x^y \overset{y}{u} = 0 \tag{2.4}$$

Premultiplying the equation (2.3) by F and using the equation (1.1),(2.2) and (2.3) itself, we get

$$a^2\underset{x}{N} = -Bf\underset{x}{U} - \sqrt{\varepsilon}\sum_{x=1}^{r} \overset{z}{u}\left(\underset{x}{U}\right)\underset{z}{N} + \sqrt{\varepsilon}\sum_{y=1}^{r} \theta_x^y - B\underset{y}{U} + \sqrt{\varepsilon}\sum_{z=1}^{r} \theta_y^z \underset{z}{N}$$

Comparison of tangential and normal vectors again gives

$$(i) f\underset{x}{U} + \sqrt{\varepsilon}\sum_{y=1}^{r} \theta_x^y \underset{y}{U} = 0$$
and
$$(ii) \overset{z}{u}\left(\underset{x}{U}\right) + \sqrt{\varepsilon}\sum_{y=1}^{r} \theta_x^y \theta_y^z = \frac{a^2}{\sqrt{\varepsilon}\delta_x^z} \tag{2.5}$$

Again in view of the equations (1.2), (2.2) and (2.3) we have

$$g(fX, fY) + a^2g(X,Y) + \varepsilon\sum_{x=1}^{r} \overset{z}{u}(X)\overset{x}{u}(Y) = 0 \tag{2.6}$$

By virtue of the equations (2.4), (2.5) and (2.6) it folllows that the submanifold M^{n-r} admits, the generalised para (ε, r)-contact metric structure. Hence we have

Theorem 1. *The submanifold M^{n-r} of codimension r of the H-structure manifold M^n admits a generalised para (ε, r)-contact metric structure.*

Suppose further that D is the induced connection on the submanifold M^{n-r} from the Riemannian connection $\widetilde{D}$ on the enveloping manifold M^n. The equations Gauss and Weingarten can be expressed as[4]

$$\widetilde{D}_{BX}BY = BD_XY + \sum_{y=1}^{r} \overset{x}{h}(X,Y)\underset{x}{N} \tag{2.7}$$

and

$$\widetilde{D}_{BX}\underset{x}{N} = -B\overset{x}{H}(X) + \sum_{y=1}^{r} \theta_x^y \underset{x}{N} \tag{2.8}$$

where $\overset{x}{h}(X,Y)$ are second fundamental forms given by

$$\overset{x}{h}(X,Y) = g\left(\overset{x}{H}(X),Y\right), x = 1,2,\dots,r \quad (2.9)$$

Suppose that the enveloping manifold M^n is a K-manifold. Hence we have

$$\left(\widetilde{D}_{BX}F(BY)\right) = 0$$

or equivalently

$$\widetilde{D}_{BX}FBY = F\widetilde{D}_{BX}BY \quad (2.10)$$

In view of the equations (2.2) and (2.7) , the above equation (2.10) takes the form

$$D_{BX}\{BfY - \sqrt{\varepsilon}\sum_{x=1}^{r}\overset{x}{u}(Y)\underset{x}{N} = F\{BD_XY + \sum_{x=1}^{r}\overset{x}{h}(X,Y)\underset{x}{N}\}$$

or equivalently

$$BD_XfY + \sum_{x=1}^{r} h(X,fY)\underset{x}{N} - \sqrt{\varepsilon}\sum_{x=1}^{r}\overset{x}{u}(Y)\{-BH(X) + \sum_{y=1}^{r}\theta_x^y \underset{y}{N}\}$$

$$= BfD_XY - \sqrt{\varepsilon}\sum_{x=1}^{r}\overset{x}{u}(D_XY)\underset{x}{N} + \sum_{x=1}^{r}\overset{x}{h}(X,Y)\{-B\underset{x}{U} + \sqrt{\varepsilon}\sum_{y=1}^{r}\theta_x^y \underset{x}{N}\}$$

Comparison of tangential vector fields yields

$$D_XfY + \sqrt{\varepsilon}\sum_{x=1}^{r}\overset{x}{u}(Y)\overset{x}{H}(X) = fD_XY - \sum_{x=1}^{r}\overset{x}{h}(X,Y)\underset{x}{U}$$

or equivalently

$$(D_Xf)(Y) + \textstyle\sum_{x=1}^{r}\{\sqrt{\varepsilon}\overset{x}{u}(Y)\overset{x}{H}(X) + \overset{x}{h}(X,Y)\underset{x}{U}\} = 0 \quad (2.11)$$

If N(X,Y) be Nijenhuis tensor for the submanifold M^{n-r}, we can write [4]

$$N(X,Y) = \left(D_{fX}f\right)(Y) - \left(D_{fY}f\right)(X) + f\left(D_yf\right)(X) - f(D_Xf)(Y) \quad (2.12)$$

A necessary and sufficient condition that the submanifold M^{n-r} be totally geodesic is that $h^x(X,Y) = 0$, x=1,2,...,r. Hence in view of the equation (2.6) and (2.8), it follows that

$$D_Xf = 0$$

Hence from the equation (2.12), it follows that

$$n(X,Y) = 0$$

Thus we have.

Theorem 2. *A totally geodesic submanifold M^{n-r} with a generalized para (ε, r)-contact structure of an H-structure manifold is integrable.*

3. Contravariant Almost Analytic Vectors

In the K-manifold M^n, taking a=i,$i = \sqrt{\varepsilon}$ and n even, we observe that M^n becomes a Kaehler manifold. It is well know that for a Kaehler manifold the contravariant almost analytic vector V^* satisfies [1]

$$F\widetilde{D}_{X^*}V^* + \widetilde{D}_{X^*}V^* = 0 \tag{3.1}$$

Hence we have.

$$F\widetilde{D}_{BX}BV + \widetilde{D}_{BX}BV = 0 \tag{3.2}$$

In view of equation (2.4) above equation (3.2) takes the form

$$F\{BD_XV + \textstyle\sum_{x=1}^{r} \overset{x}{h}(X,V)\underset{x}{N}\} + BD_XV + \textstyle\sum_{x=1}^{r} \overset{x}{h}(X,V)\underset{x}{N} = 0 \tag{3.3}$$

By virtue of the equations (2.2) and (2.3) and , the above equation (3.3) takes the form

$$BfD_xV - \sqrt{\varepsilon}\sum_{x=1}^{r} \overset{x}{u}(D_XV)\underset{x}{N} + \sum_{x=1}^{r} h(X,V)\{-BU + \sqrt{\varepsilon}\sum_{y=1}^{r} \theta_x^y \underset{y}{N}\} + BD_XV + \sum_{x=1}^{r} \overset{x}{h}(X,V)\underset{x}{N} = 0 \tag{3.4}$$

Comparison of tangential vectors fields

$$fD_XV - \textstyle\sum_{x=1}^{r} \overset{x}{h}(X,V)\underset{x}{U} + D_XV = 0 \tag{3.5}$$

We know that the necessary and sufficient condition that the submanifold M^{n-r} be totally geodesic is that

$$h(X,V) = 0, x = 1,2,\ldots,r$$

Hence the equation (3.5) becomes

$$fD_XV + D_XV = 0$$

So the vector V is contravariant almost analytic in the submanifold M^{n-r}.Thus we have.

Theorem 3. *If M^n be a Kaehler manifold and M^{n-r} its totally geodesic submanifold admitting a generalised para (ε, r)-contact structure, the contravariant almost analytic vector field in the enveloping manifold induces a similar vector field in the submanifold.*

References

[1]. Nivas, R. & Srivastava, S. K. (1996); On almost r-contact structure Indian Journal of Physical and Natural Sciences, Vol.-12(B), pp. 11-17.

[2]. Srivastava, S. K. (2000); On submanifold of a manifold admitting hyperbolic ε-structure Journal of Ravishankar University, Vol-13(B), pp. 30-38.

[3]. Nivas, R. and Ahmad, S. (1990); Submanifold of codimension r of an H-structure manifold. Rov. Mat. Univ. Parma (4) 16; pp. 167-172, Italy.

[4]. Rai, R. P. (1987); Submanifold of codimension 2 of an almost hyperbolic hermite manifold, Journal of the Indian academy of Mathematics, Vol. 9(1), pp. 1-10.

[5]. Nagarrajan, H.G. & Kumar, D. (2021); Submanifold of N(K)-contact metric manifolds Palestine Journal of Mathematics, Vol 10(2), pp. 777-783