ON $F_a(K,(K-2))$-STRUCTURE MANIFOLD

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Abstract

Yano, Houh and Chen [1] have studied the structure defined by the tenosor field ϕ of type (1,1) satisfying $\phi^4 \pm \phi^2 = 0$. Gadea and Cordero [2] have obtained the integrability conditions of these structures. The purpose of this paper is to define and study $F_a(K,(k-2))$-structure. Integrability conditions of such a structure have also been studied.
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1. $F_a(K,(K-2))$-Structure

Let M^n be an even dimensional differentiable manifold of differentiability class C^∞. Suppose there exists on M^n a tensor field F of type (1,1) and of class C^∞ satisfying

$$F^K + a^2 F^{K-2} = 0 \tag{1.1}$$

where K is an odd positive integer and 'a' is any complex member not equal to zero. Also

$$(2RankF - RankF^{K-1}) = dimM^n \tag{1.2}$$

Let us define the operators 's' and 't' on M^n as follows:

$$\begin{aligned}&(i) s = (-)^{1/2(K-1)} \frac{F^{K-1}}{a^{K-1}}\\ &and\\ &(ii) t = I - (-)^{1/2(K-1)} \frac{F^{K-1}}{a^{K-1}}\end{aligned} \tag{1.3}$$

I denotes the unit tensor field. Thus we have.

Theorem 1. *For The (1,1) tensor field F satisfying equation (1.1) the operators 's' and 't' defined by (1.3) when applied to the tangent space of M^n at a point are complementary projection operators.*

Proof.
We have from the equation (1.3).

$$s + t = I \quad (1.4)$$

Also,

$$\begin{aligned} s^2 &= (-)^{(K-1)} \frac{F^{2K-2}}{a^{2K-2}} = F^K . \frac{F^{K-2}}{a^{2K-2}} \\ &= -a^2 F^{K-2} \frac{F^{K-2}}{a^{2K-2}} = -F^K . \frac{F^{K-4}}{a^{2K-4}} \\ &= (-)^2 F^{K-2} \frac{a^2 F^{K-4}}{a^{2K-4}} = (-)^2 F^K \frac{F^{K-6}}{a^{2K-6}} \\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= (-)^{\frac{(K-1)}{2}} a^2 F^{K-2} \frac{F^{K-(K-1)}}{a^{2K-(K-1)}} = (-)^{1/2(K-1)} \frac{F^{K-1}}{a^{K-1}} \end{aligned}$$

or

$$s^2 = s \quad (1.5)$$

Also

$$\begin{aligned} t^2 &= I + (-)^{(K-1)} \frac{F^{2K-2}}{a^{2K-2}} - 2(-)^{1/2(K-1)} \frac{F^{K-1}}{a^{K-1}} \\ &= I - (-)^{1/2(K-1)} \frac{F^{K-1}}{a^{K-1}} \end{aligned}$$

or

$$t^2 = t \quad (1.6)$$

Further

$$\begin{aligned} st &= ts = (-)^{1/2(K-1)} \frac{F^{K-1}}{a^{K-1}} - (-)^{(K-1)} \frac{F^{2K-2}}{a^{2K-2}} \\ &= 0 \qquad \text{as} \qquad s^2 = s \end{aligned}$$

$$st = ts = 0 \quad (1.7)$$

The theorem follows by virtue of equation (1.4) to (1.7).
Let S and T be the complementary distributions corresponding to the projection operators 's' and 't' respectively. If the rank of F is constant every where and equal to r then $dimS = 2r - n$ and $dimT = 2n - 2r$, $n \leq 2r \leq 2n$. Obviously dimensions of S and T are also even. Let us call such a structure on M^n as $F_a\big(K, (K-2)\big)$-structure of rank r.

Theorem 2. *For a tensor field F (≠ 0) of type (1,1) satisfying (1.1) and for the operators 's' and 't' given by the equations (1.3) we have*

$$(i)\frac{F^{K-2}}{a^{K-1}}s = \frac{sF^{K-2}}{a^{K-1}} = \frac{F^{K-2}}{a^{K-1}} \quad \text{and} \quad (ii)\frac{F^{K-2}}{a^{K-1}}t = \frac{tF^{K-2}}{a^{K-1}} = 0 \tag{1.8}$$

Proof.
Proof follows easily by virtue of equations (1.1), (1.3) and (1.8).

Theorem 3. *For a tensor field F (≠ 0) satisfying the equation (1.1) and for the operators 's' and 't' given by (1.3), we have*

$$\begin{aligned}&(i)Fs = sF = -(-)^{1/2(K-1)}\frac{F^{K-2}}{a^{K-3}}\\&(ii)(F^2 + a^2)s = 0\\&(iii)F^2t - a^2s = F^2\end{aligned} \tag{1.9}$$

Proof.
Proof follows easily in a way similar to that of the theorem (2).

Theorem 4. $F_a\big(K,(K-2)\big)$*-structure of maximal rank is a GF-structure.*

Proof.
If the rank of F is maximum, $r = n$. So $dimS = n$ and $dimT = 0$.
Therefore

$$t = 0 \quad \text{and} \quad s = I$$

So

$$I - (-)^{1/2(K-1)}\frac{F^{(K-1)}}{a^{(K-1)}} = 0 \tag{1.10}$$

or

$$(-)^{1/2(K-1)}\frac{F^{(K-1)}}{a^{(K-1)}} = I \tag{1.11}$$

Operating the equation (1.10) by F^2 and making use of the equation (1.1) we get

$$F^2 - (-)^{1/2(K-1)}(-a^2)\frac{F^{(K-1)}}{a^{(K-1)}} = 0$$

which in view of the equation (1.11) takes the form

$$F^2 + a^2I = 0$$

Taking $-a = \lambda^2$, the above equation takes the form

$$F^2 = \lambda^2 I$$

Hence M^n admits a GF-structure.

2. Nijenhuis Tensor of $F_a(K,(K-2))$-structure

The Nijenhuis tensor formed with such F is given by

$$N(X,Y) = [FX,FY] = -F[FX,Y] - F[X,FY] + F^2[X,Y] \tag{2.1}$$

Since $s + t = I$ hence (2.1) takes the form

$$\begin{aligned} N(X,Y) &= [FsX + FtX, FsY + FtY] - F[FsX + FtX, sY + tY] \\ &\quad -F[sX + tX, FsY + FtY] + F^2[sX + tX, sY + tY] \\ &= \{[FsX,FsY] - F[FsX,sY] - F[sX,FsY] + F^2[sX,sY]\} \\ &\quad +\{[FsX,FtY] - F[FsX,tY] - F[sX,FtY] + F^2[sX,tY]\} \\ &\quad +\{[FtX,FsY] - F[FtX,sY] - F[tX,FsY] + F^2[tX,sY]\} \\ &\quad +\{[FtX,FtY] - F[FtX,tY] - F[tX,FtY] + F^2[tX,tY]\} \end{aligned}$$

or

$$N(X,Y) = N(sX,sY) + N(sX,tY) + N(tX,sY) + N(tX,tY) \tag{2.2}$$

If the distribution S is integrable, $N(sX,sY)$ is exactly the Nijenhuis tensor of $F/S \overset{def}{=} FS$. If the distribution T is integrable, $N(tX,tY)$ is exactly the Nijenhuis tensor $F/T \overset{def}{=} FT$.

If $L_Y F$ denotes the Lie-derivatives of the tensor-field F with repsect to a vector field Y, $L_Y F$ is the tensor-field of the same type as F. Also

$$(L_Y F)(X) = F[X,Y] - [FX,Y] \tag{2.3}$$

In view of the equations (2.1) and (2.3), we have

$$N(sX,tY) = F(L_{tY}F)(sX) = (L_{FtY}F)(sX) \tag{2.4}$$

and

$$N(tX,sY) = F(L_{sY}F)(tX) = (L_{FsY}F)(tX) \tag{2.5}$$

3. Integrability Conditions

In this section, we shall obtain the partial integrability conditions of $F_a(K,(K-2))$-structure (K odd).

Theorem 5. *For any two vector fields X and Y the following results hold:*

1. *The distribution S is integrable if and only if $tN(sX,sY) = 0$;*
2. *The distribution T is integrable if and only if $tN(tX,tY) = 0$.*

Proof.

We know that for any two vector fields X and Y, the distributions S and T are integrable if and only if $t[sX,sY] = 0$ and $s[tX,tY] = 0$ [2]. Thus in view of the equations (1.7), (1.9), (1.10) and (2.1), the proof of the theorem follows.

Theorem 6. *For any two vector fields X, Y, the distributions S and T are both integrable if and only if*

$$N(X,Y) = sN(sX,sY) + N(sX,tY) + N(tX,sY) + tN(tX,tY) \quad (3.1)$$

Proof.
In Consequence of the equations (1.4) and (2.2) we can write

$$\begin{aligned} N(X,Y) &= sN(sX,sY) + tN(sX,sY) + N(sX,tY) \\ &+N(tX,sY) + sN(tX,tY) + tN(tX,tY) \end{aligned} \quad (3.2)$$

The proof of the theorem follows by virtue of equation (3.2) and the theorem (5).

Theorem 7. *If the distribution S is integrable, a necessary and sufficient condtion for the GF-structure defined by $F/S = F_S$ on each integral manifold of S to be integrable is that for any two vectors field X an Y.*

$$N(sX,sY) = 0 \quad (3.3)$$

which is equivalent to $sN(sX,sY) = 0$

Proof.
Suppose the distribution S is integrable. Then F induces on each integral manifold of S, a GF-structure. The induced structure is integrable if and only if its Nijenhuis tensor vanishes identically. Thus the proof of this theorem follows.

Definition 1. *We say that $F_a\big(K,(K-2)\big)$-structure is 's_K-partially integrable' if the distribution S is integrbale and the GF-structure induced from F on each integral manifold of S is also integrable.*

Theorem 8. *For any two vector fields X and Y, a necessary and sufficient condtion for $F_a\big(K,(K-2)\big)$-structure to be 's_K-partially integrable' is that*

$$N(sX,sY) = 0 \quad (3.4)$$

Proof.
Proof follows easily from the theorem (5)(i) and (7).

Theorem 9. *If the distribution T is integrable, a necessary and sufficient condition for the structure defined by $F/T = F_T$ on each integral manifold of T to be integrable is that*

$$N(tX,tY) = 0 \quad (3.5)$$

for arbitrary vector fields X and Y.

Proof.
Proof follows easily in a way similar to that of the theorem (7).

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