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# OPTIMAL INVESTMENT STRATEGIES FOR DEFINED CONTRIBUTION (DC) PENSION FUND WITH MULTIPLE CONTRIBUTORS VIA LEGENDRE TRANSFORM AND DUAL THEORY

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**Abstract**

We studied in this paper the optimal investment strategies for DC pension fund with multiple contributors by applying Hamilton Jacobi – Bellman equation, Legendre transformation and dual theory to find the explicit solution for CRRA and CARA utility functions respectively. We obtained a general solution for CARA utility function similar to that with only one contributor as in Gao (2009) and a different result for CRRA utility function when compared to the one with one contributor.

**Keywords:** CRRA, CARA, DC, Pension fund, Optimal Strategies, Legendre Transform.

## 1. Introduction

Defined contribution pension is very crucial in retirement income system in a lot of countries and there is a growing trend to automatically involve all workers in it. In as much the DC scheme is relatively new compared to the defined benefit (DB) pension scheme, it forms a determining factor of the old age income adequacy for future retirees. This system underscores the need to understand better the risks that affect the income provided by this plan. There are basically four factors that determine retirement income adequacy in DC plans namely length of contribution and retirement period, level of contributions, management cost and investment strategies. This study focuses on investment strategies, since members of DC pension plan has some leverage in choosing their investment plan, they have to solve the optimal investment strategies problem to determine the best way to invest and maximize profit while reducing the risk involve. The most commonly used utility functions are the constant relative risk aversion (CRRA) see Cairns et al.(2006), Gao (2008), Boulier et al. (2001), Deelstra et al (2003), Xiao et al (2007) and constant absolute risk aversion (CARA) Battocchio and Menoncin (2004), Gao (2009)

The optimal investment for DC with stochastic interest rate have been studied by Boulier et al. (2001) and Battocchio and Menoncin (2004)] where the interest rate was Vasicek model, Chubing and Ximing (2013), Deelstra et al (2003) and Gao (2008), studied the affine interest rate which include the Cox- Ingeroll- Ross (CIR) model and Vasicek model. Recently, more attention has been given to constant elasticity of Variance (CEV) model in DC pension fund investment strategies. As Geometric Brownian motion (GBM) can be considered as a special case of the (CEV) model, such work extended the research of Xio et al (2007) where they applied (CEV) model to derive dual solution of a CRRA utility function via legendre transform, also Gao (2009) extended the work of Xiao et al (2007) by obtaining solutions for investor with CRRA and CARA utility function, recently Dawei and Jingyi (2014) extended the work of Gao (2009) by modelling pension fund with multiple contributors where benefit payment are made after retirement , he went on to find the explicit solution for CRRA and CARA using power transformation method. In this paper we solve the optimized problem via

Legendre transformation method and dual theory and compare the solution with that of Dawei and Jingyi (2014)

## 2. Mathematical Model

In a pension fund system with multiple contributors, it is expected that payment are remitted to contributors who have retired from service and the payment continues till the death of a specific contributor after which payment is stopped for that particular contributor. As stated by Dawei and Jingyi (2014) that the payment is a stochastic process and assume the Brownian motion with drift as follows

$$dC(t) = adt - bdW^{\circ}(t), \quad (1)$$

where $a$ and $b$ are positive constants and denote the amount given to the retired contributors and that which which is due death contributors which are out of the system.

Assume the market is made up of risk free asset (cash and bond) and risky asset (stock). Let $(\Omega, F, P)$ be a complete probability space where $\Omega$ is a real space and $P$ is a probability measure, $\{W^{\circ}(t), W_t(t)\}$ is a standard two dimensional motion. $F$ is the filtration and denotes the information generated by the Brownian motion.

Let $S_0(t)$ denote the price of the risk free asset, with dynamics given as

$$\frac{dS_0(t)}{S_0(t)} = rdt, \quad (2)$$

and let $S_t(t)$ denote the risky asset and its dynamics is given based on its stochastic nature and the price process described by the CEV model in Gao (2009) as

$$\frac{dS_t(t)}{S_t(t)} = \mu dt + K{S_t}^{\beta} dW_t. \quad (3)$$

where $\mu$ an expected instantaneous rate of return of the risky asset and satisfies the general condition $\mu > r_0$. $K{S_t}^{\beta}$ is the instantaneous volatility, and $\beta$ is the elasticity parameter and satisfies the general condition $\beta < 0$.

Now consider that in DC plans the contributions provided by the contributors are fixed and then without loss of generality, we assume that the number of contributors is constant and so is the contribution rate $c = (1 + \theta)a$ with safety loading $\theta > 0$. If there is no investment, the dynamics of the surplus is given by

$$dR(t) = cdt - dC(t) = \theta adt + bdW^{\circ}(t). \quad (4)$$

Let $V_t$ denote the wealth of pension fund at $t \in [0, T]$, let $\pi_t$ denote the proportion of the pension fund invested in the risky asset $S_t$ and $1 - \pi_t$, the proportion invested in risk free asset . Hence the dynamics of the pension wealth is given by

$$dV_t = \pi_t V_t \frac{dS_t(t)}{S_t(t)} + (1 - \pi_t)\frac{dS_0(t)}{S_0(t)} + \theta adt + bdW^{\circ}(t). \quad (5)$$

Substituting (2) and (3) into (5) we have

$$dV_t = [V_t(\pi_t(\mu - r) + r) + \theta a]dt + bdW^{\circ}(t) + \pi_t V_t K{S_t}^{\beta} dW_t \quad (6)$$

## 3. Optimization Problem

In this section we are interested in maximizing the utility of the plan contributor's terminal relative wealth. Let $H_{\pi_t}$ represent which is define to be the utility attained by the plan contributors from a given state $v$ at time t as

$$H_{\pi_t}(t,s,v) = E_{\pi_t}\left[U\big(V(T)\big) \mid S(t) = s, V(t) = v\right], \tag{7}$$

where $t$ is the time, $S$ is the price of the risky asset and $v$ is the wealth. Our aim is to obtain the optimal value function

$$H(t,s,v) = \sup_{\pi_t} H_{\pi_t}(t,s,v) \tag{8}$$

and the optimal strategy $\pi_t$ such that

$$H_{\pi_t}(t,s,v) = H(t,s,v). \tag{9}$$

The Jacobi Hamilton-Jacobi-Bellman (HJB) equation associated with the optimization problem is

$$H_t + \mu s H_s + (rv + \theta a)H_v + \frac{1}{2}k^2 s^{2\beta+2} H_{ss} + \frac{1}{2}b^2 H_{vv} +$$
$$\sup\left\{\frac{1}{2}\pi^2 k^2 s^{2\beta} v^2 H_{vv} + \pi\left[(\mu - r)vH_v + k^2 s^{2\beta+1} v H_{vs}\right]\right\} = 0. \tag{10}$$

To obtain the first order maximizing condition for $\pi^*$, we solve

$$\pi^2 k^2 s^{2\beta} v^2 H_{vv} + \pi\left[(\mu - r)vH_v + k^2 s^{2\beta+1} v H_{vs}\right] = 0, \tag{11}$$

so that

$$\pi^* = -\frac{\left[(\mu-r)H_v + k^2 s^{2\beta+1} H_{vs}\right]}{k^2 s^{2\beta} v H_{vv}}. \tag{12}$$

Substituting (12) into (10), we have

$$H_t + \mu s H_s + (rv + \theta a)H_v + \frac{1}{2}k^2 s^{2\beta+2} H_{ss} + \frac{1}{2}b^2 H_{vv} - \left[\frac{\left((\mu-r)H_v + k^2 s^{2\beta+1} H_{vs}\right)^2}{2k^2 s^{2\beta} H_{vv}}\right] = 0. \tag{13}$$

So that

$$H_t + \mu s H_s + (rv + \theta a)H_v + \frac{1}{2}k^2 s^{2\beta+2}\left[H_{ss} - \frac{{H_{vs}}^2}{H_{vv}}\right]$$
$$+\frac{1}{2}b^2 H_{vv} - \frac{z^2(\mu-r)^2}{2k^2 s^{2\beta}}\frac{{H_v}^2}{H_{vv}} - (\mu - r)s\frac{H_v H_{vs}}{H_{vv}} = 0. \tag{14}$$

## 4. Legendre Transformation

Since the differential equation obtained in (14) is a non linear PDE and quite complex to solve, we will employ the Legendre transform and dual theory to transform it to a linear PDE

**Theorem 4.1 [10]:** Let $f: R^n \to R$ be a convex function for $z > 0$, define the Legendre transform

$$L(z) = \max_x\{f(x) - zx\}, \tag{15}$$

where $L(z)$ is the Legendre dual of $f(x)$.

Since $f(x)$ is convex, from theorem 4.1 we can defined the Legendre transform

$$\widehat{H}(t,s,z) = \sup\{\, H(t,s,v) - zv \mid 0 < x < \infty\} \;\; 0 < t < T. \tag{16}$$

where $\widehat{H}$ is the dual of $H$ and $z > 0$ is the dual variable of $v$.

The value of $v$ where this optimum is attained is denoted by $g(t,s,z)$, so that

$$g(t,s,z) = \inf\{\, v \mid H(t,s,v) \geq zv + \widehat{H}(t,s,z)\} \;\; 0 < t < T. \tag{17}$$

The function g and $\widehat{H}$are closely related and can be refers to as the dual of H. These functions are related as follows

$$\widehat{H}(t,s,z) = H(t,s,g) - zg. \tag{18}$$

Where

$$g(t,s,z) = v, H_v = z, \qquad g = -\hat{H}_{z.}$$

At terminal time, we denote

$$\hat{U}(z) = \sup\{\, U(v) - zv \mid 0 < v < \infty \},$$

and

$$G(z) = \sup\{v \mid U(v) \geq zv + \hat{U}(z).$$

As a result

$$G(z) = (U')^{-1}(z), \tag{19}$$

where $G$ is the inverse of the marginal utility U and note that $H(T,s,v) = U(v)$

At terminal time $T$, we can define

$g(T,s,z) = \inf_{v>0}\{\, v \mid U(v) \geq zv + \hat{H}(t,s,z)\,\}\, and\, \hat{H}(t,s,z) = \sup_{v>0}\{\, U(v) - zv\}$

so that

$$g(T,s,z) = (U')^{-1}(z). \tag{20}$$

Next we differentiate (18) with respect to $t, s, and\ v$

$$H_t = \hat{H}_t,\ H_s = \hat{H}_s\,, H_v = z, H_{sv} = \frac{-\hat{H}_{sz}}{\hat{H}_{zz}}, H_{vv} = \frac{-1}{\hat{H}_{zz}}\ ,\ H_{sv} = \hat{H}_{ss} - \frac{{\hat{H}_{sz}}^2}{\hat{H}_{zz}}. \tag{21}$$

Substituting (21) into (14),we have

$$\hat{H}_t + \mu s\hat{H}_s + (rv + \theta a)z + \frac{1}{2}k^2 s^{2\beta+2}\hat{H}_{ss} - \frac{1}{2}b^2\frac{1}{\hat{H}_{zz}} - \frac{z^2(\mu-r)^2}{2k^2s^{2\beta}}{\hat{H}_{zz}}^2 - (\mu - r)sz\hat{H}_{zz} = 0 \tag{22}$$

and

$$\pi^* = -\frac{[(\mu-r)z\hat{H}_{zz} - k^2s^{2\beta+1}\hat{H}_{sz}].}{k^2s^{2\beta}v} \tag{23}$$

Differentiating (22) and (23) with respect to $z$ and using $v = g = -\hat{H}_z$, we have

$$g_t + rsg_s - (rg + \theta a) + \frac{1}{2}k^2s^{2\beta+2}g_{ss} + \left(\frac{(\mu-r)^2}{k^2s^{2\beta}} - r\right)zg_z$$

$$+\frac{1}{2}b^2\frac{g_{zz}}{{g_z}^2} + \frac{z^2(\mu-r)^2 g_{zz}}{2k^2s^{2\beta}} - (\mu - r)szg_{sz} = 0 \tag{24}$$

and

$$\pi^* = -\frac{[(\mu-r)zg_z - k^2s^{2\beta+1}g_s]}{gk^2s^{2\beta}}. \tag{25}$$

## 5. Optimal investment strategy for specific utility

In this section we give the explicit solution for the CRRA and CARA utility functions.

Assume the plan contributor takes a power utility function

$$U(x) = \frac{x^p}{p},\quad p < 1,\ p \neq 0 \tag{26}$$

The relative risk aversion of a decision maker with utility described by the above equation is constant and (26) is a CRRA utility.

**Proposition 5.1:** The relative risk aversion of a decision maker with CRRA utility is not constant at different time point for different price of risky asset.

Proof:

Assume the relative risk aversion of a decision maker with CRRA utility is constant regardless of changes in t and s.

We conjecture a solution to (26) with the following form

$$g(t,s,z) = h(t,s)\left[z^{\frac{1}{p-1}}\right] + a(t), \qquad a(T) = 0, \qquad h(T,s) = 1.$$

Then

$$g_t = h_t z^{\frac{1}{p-1}} + a', \; g_z = -\frac{h}{1-p} z^{\left(\frac{1}{p-1}-1\right)}, \; g_{sz} = -\frac{h_s}{1-p} z^{\left(\frac{1}{p-1}-1\right)},$$

$$g_{zz} = \frac{(2-p)h}{(1-p)^2} z^{\left(\frac{1}{p-1}-1\right)}, g_s = h_s z^{\frac{1}{p-1}}, g_{ss} = h_{ss} z^{\frac{1}{p-1}}. \tag{27}$$

Substitute (27) into (24) we have

$$\left[h_t + rsh_s - rh + \frac{1}{2}k^2 s^{2\beta+2} h_{ss} - \frac{h}{1-p}\left(\frac{(\mu-r)^2}{k^2 s^{2\beta}} - r\right) + \frac{(\mu-r)^2}{2k^2 s^{2\beta}} \frac{(2-p)h}{(1-p)^2} - (\mu-r)s\frac{h_s}{1-p}\right] z^{\frac{1}{p-1}} +$$
$$\left(\frac{1}{2}b^2 \frac{h}{2-p}\right) z^{\frac{-1}{p-1}} + \left(a' + ar - \theta a\right) = 0. \tag{28}$$

From (28) we have

$$h_t + rsh_s - rh + \frac{1}{2}k^2 s^{2\beta+2} h_{ss} - \frac{h}{1-p}\left(\frac{(\mu-r)^2}{k^2 s^{2\beta}} - r\right) + \frac{(\mu-r)^2}{2k^2 s^{2\beta}} \frac{(2-p)h}{(1-p)^2} - (\mu-r)s\frac{h_s}{1-p} = 0 \tag{29}$$

So that

$$\frac{1}{2}b^2 \frac{h}{2-p} = 0 \tag{30}$$

and

$$a' + ar - \theta a = 0. \tag{31}$$

From (30), since $b > 0$ we can see that the earlier assumption that $h(T,s) = 1$, is a contradiction. Hence the relative risk aversion of a decision maker with CRRA utility is not constant at different time point for different price of risky asset.

Assume the contributor takes an exponential utility

$$U(x) = -\frac{1}{q}e^{-qx}, \quad q > 0. \tag{32}$$

The absolute risk aversion of a decision maker with the utility described in (32) is constant and is a CARA utility

Since $g(T,s,z) = (U')^{-1}(z)$ with the CARA utility function we obtain

$$g(T,s,z) = -\frac{1}{q}lnz. \tag{33}$$

Hence we conjecture a solution to (24) with the following form

$$g(t,s,z) = -\frac{1}{q}[y(t)(lnz + m(t,s))] + w(t), \tag{34}$$

with boundary conditions $y(T) = 1,\; w(T) = 0, m(T,s) = 0$

$$g_t = -\frac{1}{q}[y'(t)(lnz + m(t,s)) + um_t] + w'(t),$$

$$g_s = -\frac{1}{q} y m_s,\ g_z = -\frac{y}{qz}\ ,\ g_{zz} = \frac{y}{qz^2}, g_{ss} = -\frac{1}{q} y m_{ss},\ g_{sz} = 0. \quad (35)$$

Substituting (35) into (24), we have

$$[y'(t) - ry(t)]lnz + [-w'(t) + rw(t) + \theta a]q +$$
$$[m_t + rsm_s + \frac{1}{2}k^2 s^{2\beta+2} m_{ss} + \frac{(\mu-r)^2}{2k^2 s^{2\beta}} - rm + \frac{y'}{y} m - r - \frac{1}{2}b^2]y = 0.$$

Such that

$$y'(t) - ry(t) = 0 \quad (36)$$

and

$$m_t + rsm_s + \frac{1}{2}k^2 s^{2\beta+2} m_{ss} + \frac{(\mu-r)^2}{2k^2 s^{2\beta}} - r - \frac{1}{2}b^2 = 0. \quad (37)$$

So that

$$-w'(t) + rw(t) + \theta a = 0 \quad (38)$$

Solving (36) and (38), we have

$$y(t) = e^{-r(t-T)} \quad (39)$$

and

$$w(t) = -\frac{\theta a}{r}(1 - e^{-r(t-T)}). \quad (40)$$

Next we conjecture a solution of (37) with the following structure

$$m(t,s) = F(t) + H(t)s^{-2\beta},\ F(T) = 0, H(T) = 0, \quad m_t = F_t + H_t s^{-2\beta}, m_s = -2\beta H s^{-2\beta-1}, m_{ss} = 2\beta(2\beta+1)Hs^{-2\beta-2}. \quad (41)$$

Substituting (41) into (37) we have

$$F_t + \beta(2\beta+1)k^2 H - r - \frac{1}{2}b^2 + s^{-2\beta}\left[H_t - 2r\beta H + \frac{(\mu-r)^2}{2k^2}\right] = 0, \quad (42)$$

so that

$$F_t + \beta(2\beta+1)k^2 H - r - \frac{1}{2}b^2 = 0, \quad (43)$$

and

$$H_t - 2r\beta H + \frac{(\mu-r)^2}{2k^2} = 0. \quad (44)$$

Solving (44) with the given condition gives;

$$H(t) = \frac{(\mu-r)^2}{4k^2 r\beta}\left[1 - e^{2r\beta(t-T)}\right]. \quad (45)$$

Next substituting (45) into (43) and solving (43) with the given condition we have

$$F(t) = \left[\frac{(2\beta+1)(\mu-r)^2}{4r} - r - \frac{1}{2}b^2\right](T-t) - \left[\frac{(2\beta+1)(\mu-r)^2}{8r^2\beta}\left(1 - e^{2r\beta(t-T)}\right)\right]. \quad (46)$$

Hence the solution to (24) for CARA utility function is given as

$$g_t(s,t,z) = -\frac{1}{q}\left[y(t)\left(lnz + m(t,s)\right)\right] + w(t),$$
$$y(t) = e^{r(t-T)},$$

$$w(t) = -\frac{\theta a\left[1 - e^{-r(T-t)}\right]}{r},$$

and

$$m(t,s) = \left[\frac{(2\beta+1)(\mu-r)^2}{4r} - r - \frac{1}{2}b^2\right](T-t) - \left[\frac{(2\beta+1)(\mu-r)^2}{8r^2\beta}\left(1 - e^{2r\beta(t-T)}\right)\right]$$
$$+ \left[\frac{s^{-2\beta}(\mu-r)^2}{4k^2 r\beta}\left(1 - e^{2r\beta(t-T)}\right)\right]. \quad (47)$$

The optimal investment strategy is given as

$$\pi^* = -\frac{\left[(\mu-r)zg_z - k^2 s^{2\beta+1} g_s\right]}{gk^2 s^{2\beta}}$$

with

$$g_t = -\frac{1}{qz}e^{r(t-T)}. \quad (48)$$

$$g_s = \frac{1}{q}e^{r(t-T)}\frac{s^{-2\beta-1}(\mu-r)^2}{2k^2 r}\left(1 - e^{2r\beta(t-T)}\right)], \quad (49)$$

and

$$\pi^* = \frac{1}{q}\frac{(\mu-r)}{k^2 s^{2\beta} g}e^{r(t-T)}\left[1 + \frac{(\mu-r)}{2r}\left(1 - e^{2r\beta(t-T)}\right)\right]. \quad (50)$$

If $ks^{\beta} = \sigma_*$ , $M(\sigma_*) = \frac{(\mu-r)}{q{\sigma_*}^2}$ , $N(t) = 1 + \frac{(\mu-r)}{2r}\left(1 - e^{2r\beta(t-T)}\right)$ *and* $g = v$,

then

$$\pi^* = g^{-1}\, M(\sigma_*)\, N(t). \quad (51)$$

Which is the same result obtained in Dawei and Jingyi (2014) when it was solved using power transformation method. Thus our result in (30) generalizes the result in Dawei and Jingyi (2014)

**Corollary 5.1**

$N(t)$ is monotonic increasing with respect to time and satisfies the condition

$$1 + \frac{(\mu-r)}{2r}\left(1 - e^{-2r\beta T}\right) \le N(t) \le 1.$$

Proof

Let $N(t) = 1 + \frac{(\mu-r)}{2r}\left(1 - e^{2r\beta(t-T)}\right)$ , $\mu - r > 0, \beta < 0$, then

$$N'(t) = -\beta(\mu - r)\left(1 - e^{-2r\beta(t-T)}\right)$$

This implies that $N(t)$ is monotonic increasing function.

When $t = 0$,

$$N(t) = 1 + \frac{(\mu - r)}{2r}\left(1 - e^{-2r\beta T}\right).$$

When $t = 1$ and $T = 0$,

$$N(t) = 1.$$

Therefore

$$1+\frac{(\mu-r)}{2r}\left(1-e^{-2r\beta T}\right)\leq N(t)\leq 1.$$

**Proposition 5.2:**

For an investor with CARA utility function, the optimal investment strategy is given as

$$\pi^*=\begin{cases} g^{-1}\,M(\sigma_*)\,N(t), & 0\leq\pi^*\leq 1\\ 1, & \pi^*>1\end{cases}.$$

Let $V$ denote the invested capital, we assume that for any $L>0$ that $V>L$, also

$$M(\sigma_*)=\frac{(\mu-r)}{k{\sigma_*}^2}>0.$$

We consider the following cases

**Case1**

If

$$\frac{(\mu-r)}{2r}\left(1-e^{-2r\beta T}\right)\geq -1\ and\ Lk{\sigma_*}^2<(\mu-r)\leq\frac{Lk{\sigma_*}^2e^{rT}}{N(0)}$$

and

$$t>T-(\ln(\mu-r)+\ln\big(N(0)\big)-\ln(Lk{\sigma_*}^2),$$

then

$$\pi^*>1.$$

From proposition 5.2, $\pi^*=1$. The implication of this is that the investor invest in only risky asset

**Case 2**

If

$$\frac{(\mu-r)}{2r}\left(1-e^{-2r\beta T}\right)<-1, t_0\leq t\leq T\ and\ Lk{\sigma_*}^2<(\mu-r)\leq\frac{Lk{\sigma_*}^2e^{r_0T}}{N(0)}$$

and

$$t>T-\frac{\ln(\mu-r)-\ln(Lk{\sigma_*}^2).}{\tau}$$

Then $0\leq\pi^*\leq 1$. From the proposition 5.2, $\pi^*=g^{-1}\,M(\sigma_*)\,N(t)$

The implication is that the investor invests the proportion equal to $g^{-1}\,M(\sigma_*)\,N(t)$ in the risky asset.

## 6. Conclusion

We have applied the Legendre transformation method and dual theory to solve the optimized problem for the optimal investment strategy for DC pension fund with multiple contributors which was solved using power transformation method by Dawei and Jingyi (2014) and we have shown that solution obtained via power transformation can as well be obtained using Legendre transform. In fact the solution via Legendre transform generalised that via power transform. Also we observed that the optimal investment strategy with only one contributor for CRRA is different from that of multiple contributors while that of CARA is the same with one contributor.

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