A New Sub Class of Univalent Analytic Functions Involving a Linear Operator

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Abstract

This paper deals with a new class T (α, β, a, b; c) that is a subclass of uniformly starlikefunctions involving a linear operator L (a, b; c). Coefficients inequality, Distortion theorem, Extreme points, Radius of starlikeness and radius of convexity for functions belonging to this class are obtained.

Key words: Univalent, starlike, convex, analytic, linear operator.

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1. Introduction

Let T denote the family of functions of the form

$$(1.1)\, f(z) = a_1 z - \sum_{n=2}^{\infty} a_{n+k} z^{n+k}, (a_1 \geq 0, a_{n+k} \geq 0, k = 0,1,2,....)$$

Which are analytic in the unit open disk Δ={z: |z|<1}.

The Hadamard product or convolution product of function $f(z) \in T$ and

$g(z) = z + \sum_{n=2}^{\infty} b_{n+k} z^{n+k}, b_{n+k} \geq 0$ is defined as

$$(1.2) \qquad (f * g)(z) = a_1 z - \sum_{n=2}^{\infty} a_{n+k} b_{n+k} z^{n+k}$$

Now, define a function $\phi(a,b;c;z)$ as

$$(1.3) \quad \phi(a,b;c;z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}} z^{n+k} forc \neq 0,-1,....,a,b \neq -1, z \in \Delta.$$

Where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(1.4)\, (\lambda)_n = \frac{\Gamma(n+\lambda)}{\Gamma(\lambda)} = \begin{cases} 1, n = 0 \\ \lambda(\lambda+1)...(\lambda+n-1), n \in N. \end{cases}$$

Now we introduced a linear operator L(a,b;c) which is defined as

$$L(a,b;c) f(z) = \phi(a,b;c;z) * f(z)$$

Thus for $f(z) \in T$

$$(1.5) L(a,b;c:z)f(z) = a_1 z - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}} a_{n+k} z^{n+k}, k = 0,1,..., z \in \Delta.$$

For b=1, the operator L(a,b;c) reduces toL(a;c) which was introduced and studied by Carlson & Shaffer[1].

We note that L(a,1;a)f(z) = f(z), L(2,1;1)f(z) = zf$'(z)$, L(m+1,1;1)f(z)=$D^m f(z)$,where $D^m f(z)$ is the Ruscheweyh (Ruscheweyh, 1975), as

$$(1.6) \qquad D^m f(z) = \frac{z}{(1-z)^{m+1}} * f(z), m > -1.$$

This is equivalently

$$D^m f(z) = \frac{z}{m!} \frac{d^m}{dz^m} \{z^{m-1} f(z)\}$$

For β≥0 and -1≤ α <1, we introduced a subclass T (α,β,a,b;c;z) of T consisting of functions f(z) of the form(1.1) and satisfying the condition

$$\mathrm{Re}\left[\frac{z\{L(a,b;c;z)f(z)\}'}{L(a,b;c;z)f(z)} - \alpha\right] > \beta\left|\frac{z\{L(a,b;c;z)f(z)\}'}{L(a,b;c;z)f(z)} - 1\right|, z \in \Delta.$$

For a_1 =1,b=1,k=0,T (α,β,a,b;c;z) reduces to TS (α,β,) which was defined and studied by G. Murugusundaramoorthy (Murugusundaramoorthy et al., 2004).

The main object of this paper is to obtain necessary and sufficient conditions for the functions f(z) ∈ T (α,β,a,b;c;z). Furthermore we obtain extreme points, distortion bounds, Closure properties, radius of starlikeness and convexity for f(z) ∈ T (α,β,a,b;c;z).

2. Coefficients Inequality

Theorem2.1: A necessary and sufficient condition for f(z) of the form (1.1) to be in the class T (α,β,a,b;c;z), -1≤ α <1, β ≥ 0 is that

$$(2.1) \sum_{n=2}^{\infty} [(n+k)(1+\beta) - (\alpha+\beta)] \frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}} |a_{n+k}| \le (1-\alpha)a_1.$$

Proof: Let f (z) ∈ T (α,β,a,b;c;z),then it is sufficient to show that

$$\beta\left|\frac{z\{L(a,b;c;z)f(z)\}'}{L(a,b;c;z)f(z)} - 1\right| - \mathrm{Re}\left[\frac{z\{L(a,b;c;z)f(z)\}'}{L(a,b;c;z)f(z)} - 1\right] \le 1-\alpha.$$

We have

$$\beta\left|\frac{z\{L(a,b;c;z)f(z)\}'}{L(a,b;c;z)f(z)} - 1\right| - \mathrm{Re}\left[\frac{z\{L(a,b;c;z)f(z)\}'}{L(a,b;c;z)f(z)} - 1\right]$$

$$\leq (1+\beta)\left|\frac{z\{L(a,b;c;z)f(z)\}'}{L(a,b;c;z)f(z)}-1\right|$$

$$\leq (1+\beta)\left|\frac{a_1 z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}(n+k)a_{n+k}z^{n+k}}{a_1 z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}a_{n+k}z^{n+k}}-1\right|$$

$$\leq (1+\beta)\frac{\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}(n+k-1)|a_{n+k}|}{a_1-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}|a_{n+k}|}$$

This expression is bounded above by $(1-\alpha)$ if

$$(1+\beta)\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}(n+k-1)|a_{n+k}|\leq a_1(1-\alpha)-(1-\alpha)\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}|a_{n+k}|$$

Or $\sum_{n=2}^{\infty}[(n+k)(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}|a_{n+k}|\leq (1-\alpha)a_1.$

Conversely let (2.1) holds. Using the fact that Re $(\omega)>\delta$ if and only if $|\omega-(1+\delta)|<|\omega+(1-\delta)|$, it is enough to show that

$$\left|\frac{z\{L(a,b;c;z)f(z)\}'}{L(a,b;c;z)f(z)}-\left(1+\beta\left|\frac{z\{L(a,b;c;z)f(z)\}'}{L(a,b;c;z)f(z)}-1\right|+\alpha\right)\right|$$

$$<\left|\frac{z\{L(a,b;c;z)f(z)\}'}{L(a,b;c;z)f(z)}+\left(1-\beta\left|\frac{z\{L(a,b;c;z)f(z)\}'}{L(a,b;c;z)f(z)}-1\right|-\alpha\right)\right|$$

Let $E=\left|\frac{z\{L(a,b;c;z)f(z)\}'}{L(a,b;c;z)f(z)}+\left(1-\beta\left|\frac{z\{L(a,b;c;z)f(z)\}'}{L(a,b;c;z)f(z)}-1\right|-\alpha\right)\right|$

$$=\left|\frac{a_1 z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}(n+k)a_{n+k}z^{n+k}}{a_1 z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}a_{n+k}z^{n+k}}+\left(1-\beta\left|\frac{a_1 z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}(n+k-1)a_{n+k}z^{n+k}}{a_1 z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}a_{n+k}z^{n+k}}\right|-\alpha\right)\right|$$

$$= \frac{1}{|L(a,b;c;z)f(z)|}\left| \begin{array}{l} a_1 z - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}(n+k)a_{n+k}z^{n+k} + (1-\alpha)a_1 z - \\ (1-\alpha)\sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}} a_{n+k}z^{n+k} - \beta a_1 z - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}(n+k-1)a_{n+k}z^{n+k} \end{array} \right|$$

Thus

$$(2.2) E > \frac{|z|}{|L(a,b;c;z)|}\left[(2-\alpha)a_1 - \sum_{n=2}^{\infty}\{(n+k+1-\alpha)+\beta(n+k-1)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}(a)_{n+k}\right]$$

Again let $F = \left| \frac{z\{L(a,b;c;z)f(z)\}'}{L(a,b;c;z)f(z)} - \left(1+\beta\left|\frac{z\{L(a,b;c;z)f(z)\}'}{L(a,b;c;z)f(z)} - 1\right| + \alpha\right)\right|$

$$= \frac{1}{|L(a,b;c;z)f(z)|}\left| \begin{array}{l} a_1 z - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}(n+k)a_{n+k}z^{n+k} - (1+\alpha)a_1 z \\ + (1+\alpha)\sum_{n=2}^{\alpha} \frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}} a_{n+k}z^{n+k} - \beta\left| - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}(n+k-1)a_{n+k}z^{n+k} \right| \end{array} \right|$$

Thus

$$(2.3) F < \frac{|z|}{|L(a,b;c;z)f(z)|}\left[\alpha a_1 + \sum_{n=2}^{\infty}\{(n+k-1-\alpha)+\beta(n+k-1)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}a_{n+k}\right]$$

Now, from (2.2), (2.3), it follows that

$$(2.4) E - F > \frac{2|z|}{|L(a,b;c;z)f(z)|}\left[(1-\alpha)a_1 - \sum_{n=2}^{\infty}\{(n+k)(1+\beta)-(\alpha+\beta)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}a_{n+k}\right]$$

Thus (2.1) proves the theorem.

The result is sharp. The extremal function being

$$(2.5)\quad f(z) = \frac{\{(n+k)(1+\beta)-(\alpha+\beta)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}} z - (1-\alpha)z^{n+k}}{\{(n+k)(1+\beta)-(\alpha+\beta)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}}, n \geq 2.$$

Corollary 2.2: Let the function f(z) defined by(1.1) be in the class T(α,β,a,b;c;z).Then

$$(2.6)\quad a_{n+k} \leq \frac{(1-\alpha)a_1}{\{(n+k)(1+\beta)-(\alpha+\beta)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}}, n \geq 2.$$

Corollary 2.3: If f (z) ∈ T (α,β,a,b;c;z),thenfor any c>-1, the function g(z)defined as

(2.7) $$g(z)=\frac{c+1}{z^{c}}\int_{0}^{z}t^{c-1}f(t)dt$$

Also belong to T (α,β,a,b;c;z).

Proof: From (2.7) it follows that

$$g(z)=a_1 z-\sum_{n=2}^{\infty}\left(\frac{c+1}{c+n+k}\right)a_{n+k}z^{n+k}.$$

Then (2.1) yields the result.

Remarks: (i) $T(\alpha_2,\beta,a,b;c;z)\subset T(\alpha_1,\beta,a,b;c;z)$ for $0\le\alpha_1<\alpha_2<1,\beta\ge 0$

(ii) $T(\alpha,\beta_2,a,b;c;z)\subset T(\alpha,\beta_1,a,b;c;z)$ for $\beta_2>\beta_1>0, 0\le\alpha<1$.

3. Distortion Theorems

Theorem3.1: If f (z) ∈ T (α,β,a,b;c;z), then for $z\in\Delta$

(3.1) $$a_1\left(|z|-\frac{(1-\alpha)c}{\{2-\alpha+\beta+k(1+\beta)\}ab}|z|^{k+2}\right)\le|f(z)|\le a_1\left(|z|+\frac{(1-\alpha)c}{\{2-\alpha+\beta+k(1+\beta)\}ab}|z|^{k+2}\right)$$

and

(3.2) $$a_1\left(|z|-\frac{(1-\alpha)}{\{2-\alpha+\beta+k(1+\beta)\}}|z|^{k+2}\right)\le|L(a,b;c)f(z)|\le a_1\left(|z|+\frac{(1-\alpha)}{\{2-\alpha+\beta+k(1+\beta)\}}|z|^{k+2}\right)$$

Proof: In view of inequality (2.1), it follows that $\sum_{n=2}^{\infty}[(n+k)(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}a_{n+k}\le(1-\alpha)a_1.$

By the fact that $\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}}$ is non-decreasing for n $\ge$ 2.Then

$$\{2-\alpha+\beta+k(1+\beta)\}\frac{ab}{c}\sum_{n=2}^{\infty}a_{n+k}\le\sum_{n=2}^{\infty}[(n+k)(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}a_{n+k}$$

$\le(1-\alpha)a_1.$

Or, $$\sum_{n=2}^{\infty}a_{n+k}\le\frac{(1-\alpha)ca_1}{\{2-\alpha+\beta+k(1+\beta)\}ab}$$

Therefore

(3.3) $$|f(z)|\ge a_1\left(|z|-\frac{(1-\alpha)c}{\{2-\alpha+\beta+k(1+\beta)\}ab}|z|^{k+2}\right)$$

and

$$(3.4)\left|f(z)\right| \le a_1\left(|z| + \frac{(1-\alpha)c}{\{2-\alpha+\beta+k(1+\beta)\}ab}|z|^{k+2}\right)$$

From (3.3) and (3.4) inequality (3.1) follows.

Further, for f (z) ∈ T (α,β,a,b;c;z),inequality (2.1) gives

$$\{2-\alpha+\beta+k(1+\beta)\}\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}a_{n+k} \le (1-\alpha)a_1.$$

$$\text{Or, } \sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}a_{n+k} \le \frac{(1-\alpha)a_1}{\{2-\alpha+\beta+k(1+\beta)\}}$$

Thus,

$$(3.5)\quad \left|L(a,b;c)f(z)\right| \ge a_1\left(|z| - \frac{(1-\alpha)}{\{2-\alpha+\beta+k(1+\beta)\}}|z|^{k+2}\right)$$

and

$$(3.6)\left|L(a,b;c)f(z)\right| \le a_1\left(|z| + \frac{(1-\alpha)}{\{2-\alpha+\beta+k(1+\beta)\}}|z|^{k+2}\right)$$

On using (3.5) and (3.6) inequality (3.2) follows.

Remark3.2: The bounds in (3.1) & (3.2) are sharp, since the inequalities are attained for the function.

$$(3.7)\quad f(z) = \frac{\{2-\alpha+\beta+k(1+\beta)\}abz-(1-\alpha)cz^{k+2}}{\{2-\alpha+\beta+k(1+\beta)\}ab}, \text{ where } 0 \le \lambda \le 1.$$

Corollary3.3: Let f (z) ∈ T (α,β,a,b;c;z),then by disk Δ is mapped on to a domain that contains a disk of radius $a_1\left[\frac{\{2-\alpha+\beta+k(1+\beta)\}ab-(1-\alpha)c}{\{2-\alpha+\beta+k(1+\beta)\}ab}\right]$

and by $L(a,b;c)f(z)$, the disk Δ is mapped on to a domain that contain a disk of radius $a_1\left[\frac{\{3-2\alpha+\beta+k(1+\beta)\}}{\{2-\alpha+\beta+k(1+\beta)\}}\right]$..

The extremal function given by (3.7) shows the sharpness of these results.

4. Extreme Points

Theorem 4.1: Let

$$(4.1)\quad f_1(z) = a_1 z \text{ and } f_n(z) = a_1 z - \frac{(1-\alpha)a_1}{\{(n+k)(1+\beta)-(\alpha+\beta)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}}z^{n+k}$$

for n ≥ 2, k = 0,1,2……,then f (z) ∈ T (α,β,a,b;c;z), if and only if it can be expressed in the form

(4.2) $f(z)=\sum_{n=1}^{\infty} d_n f_n(z)$, where $d_n \geq 0$ and $\sum_{n=1}^{\infty} d_n = 1$.

In particular the extreme points of T (α,β,a,b;c;z) are the functions given by (4.1).

Proof: Let f(z) be expressed in the form (4.1),then

$$f(z)=\sum_{n=1}^{\infty} d_n f_n(z) = a_1 z - \sum_{n=2}^{\infty} \frac{(1-\alpha)a_1 d_n}{\{(n+k)(1+\beta)-(\alpha+\beta)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}} z^{n+k}$$

$$= a_1 z - \sum_{n=2}^{\infty} d_n t_{n+k} z^{n+k}$$

Where $t_{n+k} = \dfrac{(1-\alpha)a_1}{\{(n+k)(1+\beta)-(\alpha+\beta)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}}$

Now, since

$$\sum_{n=2}^{\infty} \{(n+k)(1+\beta)-(\alpha+\beta)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}} d_n t_{n+k} = \sum_{n=2}^{\infty}(1-\alpha)a_1 d_n$$

$$= (1-\alpha)(1-d_1)a_1 \leq (1-\alpha)a_1.$$

Therefore, f (z) ∈ T (α,β,a,b;c;z).

Conversely, let f (z) ∈ T (α,β,a,b;c;z),then(2.1) yields

$$a_{n+k} \leq \frac{(1-\alpha)a_1}{\{(n+k)(1+\beta)-(\alpha+\beta)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}} z^{n+k} \text{ for } n \geq 2.$$

Setting $d_n = \dfrac{\{(n+k)(1+\beta)-(\alpha+\beta)(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}(1-\alpha)a_1} a_{n+k}$ for $n \geq 2$

and $d_1 = 1 - \sum_{n=2}^{\infty} d_n$.

Then $f(z) = a_1 z - \sum_{n=2}^{\infty} \dfrac{(1-\alpha)a_1}{\{(n+k)(1+\beta)-(\alpha+\beta)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}} d_n z^{n+k}$

$$= a_1 z - \sum_{n=2}^{\infty} d_n \{a_1 z - f_n(z)\}$$

$$= a_1 z(1 - \sum_{n=2}^{\infty} d_n) + \sum_{n=2}^{\infty} d_n f_n(z) = \sum_{n=1}^{\infty} d_n f_n(z).$$

This completes the proof.

5. Radius of Starlikeness

Theorem 5.1: Let f (z) ∈ T (α,β,a,b;c;z),then f(z) is starlike in $|z| < r(\alpha, \beta, a, b; c), where$

$$(5.1)\ r = \inf\left[\frac{\{(n+k)(1+\beta) - (\alpha+\beta)(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}(1-\alpha)(n+k)}\right]^{\frac{1}{n+k-1}}, n \geq 2, k = 0,1,2..$$

Proof: It suffices to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1$$

$$\text{i.e.,} \left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{n=2}^{\infty}(n+k-1)a_{n+k}|z|^{n+k-1}}{a_1 - \sum_{n=2}^{\infty} a_{n+k}|z|^{n+k-1}} < 1$$

$$(5.2)\ \text{or}\ \sum_{n=2}^{\infty}(n+k)a_{n+k}|z|^{n+k-1} < a_{1.}$$

It is easily to see that (5.1) holds if

$$|z|^{n+k-1} < \left[\frac{\{(n+k)(1+\beta) - (\alpha+\beta)(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}(1-\alpha)(n+k)}\right].$$

This completes the proof.

6. Radius of Convexity

Theorem 6.1: Let f (z) ∈ T (α,β,a,b;c;z),then f(z) is convex in $|z| < r(\alpha, \beta, a, b; c), where$

$$(6.1)\quad r = \inf\left[\frac{\{(n+k)(1+\beta) - (\alpha+\beta)(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}(1-\alpha)(n+k)^2}\right]^{\frac{1}{n+k-1}}, n \geq 2, k = 0,1,2.$$

Proof: Upon noting the fact that f(z) is convex if and only if $zf'(z)$is starlike, the Theorem(6.1) follows.

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