A NEW DISTRIBUTION — L PROBABILITY DISTRIBUTION FUNCTION

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Abstract

During study of problem for geostrophic and static equilibrium in atmosphere L distribution function was deduced. L function possesses unique parameter θ_M, Comparing with other famous probability distribution L distribution function has similarly certain properties such as:

(1) Variance is $(\theta_M/3)^2$; standard variance is $\theta_M/3$, mathematical expectation equal to zero,

(2) Fourth moment $(\theta_M)^4/25$, coefficient of kurtosis is 0.24, which more 0.24 than that of Normal distribution function. Third moment and coefficient of skew are both zero.

(3) m-th moment exist, probability is equal to2/e (74.04％) within coverage of $(-\theta_M/e < \theta \leq \theta_M/e)$; probability approximately 70.04％ within coverage of $(-\theta_M/3<\theta\leq \theta_M/3)$;

(4) Continuous random variables of L function thickly more scatter in area near to its mean value than Normal distribution function does.

Keywords: L probability density function; Variance; Expected value; The coefficient of kurtosis; ***m***-th moment.

1. Probability Function

1.1 Probability Density Function

Assumed some physical system to be falling into L distribution in damped physical system, L probability density function may be postulated as below;

$$f(\theta)=\frac{1}{4\theta_M}\ln(\frac{\theta_M}{\theta})^2 \qquad (-\theta_M < \theta < \theta_M) \qquad (1)$$

Here, θ is the random variable, $\boldsymbol{\theta_M}$ is initial value and maximum amplitude when $t=0$ and also unique parameter in the function, as θ *is* approaching ***0*** ,L function has the odd singularity point, however, any its integral function tends to convergence, shown as Fig 1.

1.2 *m*-th moment of L function

When ***m*** is denoted even

$$\nu_m = \frac{1}{4\theta_M}\int_{-\theta_M}^{\theta_M} \theta^m \ln(\frac{\theta_M}{\theta})^2 \mathrm{d}\theta = \frac{\theta_M^m}{(m+1)^2}$$

m=2（variance）, m=4（fourth moment）

When m is denoted odd

$$\nu_m = \frac{1}{4\theta_M}\int_{-\theta_M}^{\theta_M} \theta^m \ln(\frac{\theta_M}{\theta})^2 \mathrm{d}\theta = 0$$

m=1（expected value）, *m*=3（third moment), respectively.

In proof of above, below can be certified firstly by using L' Hopital's Rule

$$\lim_{\theta\to 0}\frac{2}{m+1}\theta^{m+1}\ln\theta \to$$

$$\frac{2}{(m+1)}\frac{\lim\limits_{\theta\to 0}\ln\theta}{\lim\limits_{\theta\to 0}\frac{1}{\theta^{m+1}}} \to 0$$

1.3 The Distribution Function

We integrate formula (1), L probability density function, and then distribution function is verified below

$$F(\theta) = \frac{1}{4\theta_M}\{[\theta\ln(\frac{\theta_M}{\theta})^2 + 2\theta] + 2\theta_M\}$$

$$(-\theta_M < \theta < \theta_M)$$

Certainly $F'(\theta) = f(\theta)$

1.4 The how many of probability in any interval (θ_1 , θ_2）

According to the theorem

$$P(a < Y < b) = F(b) - F(a), -\infty < a \le b < \infty$$

Therefore the probability of any interval （θ_1 , θ_2）for L distribution is calculated via below formula

$$P(\theta_1 < \theta < \theta_2) = F(\theta_2) - F(\theta_1)$$
$$= \frac{1}{4\theta_M}\int_{\theta_1}^{\theta_2} \ln(\frac{\theta_M}{\theta})^2 \, d\theta$$
$$= \frac{1}{4\theta_M}\{[\theta_2 \ln(\frac{\theta_M}{\theta_2})^2 + 2\theta_2] - [\theta_1 \ln(\frac{\theta_M}{\theta_1})^2 + 2\theta_1]\}$$
$$(-\theta_M < \theta < \theta_M)$$

For example:

P (θ_1<θ<θ_2) =2/e (approximately 74.04%), If θ_1=-θ_M/e, θ_2=θ_M/e.

Here e =2.718281828459, obviously θ_M/e is very closer to (θ_M/3), comparatively, the probability. The Normal distribution lies in the interval between ±σ（its standard deviation), is 68.3% or so, but the probability of L distribution is nearly exact 70.0% in same interval. In addition the limit of formula above exists when θ_1→0,θ_2→0, shown as Fig 1 and Fig 2.

1.5 The exits of distribution function when closer to zero

$$\lim_{\theta\to 0} F(\theta) = \lim_{\theta\to 0} \ln(\frac{\theta_M}{\theta})^{\frac{\theta}{2\theta_M}} + \frac{\theta}{2\theta_M} \Big|_{\theta\to 0} + \frac{1}{2}$$
$$= \lim_{\theta\to 0} \ln(\theta_M)^{\frac{\theta}{2\theta_M}} - \lim_{\theta\to 0} \ln(\theta)^{\frac{\theta}{2\theta_M}} + \frac{\theta}{2\theta_M}\Big|_{\theta\to 0} + \frac{1}{2}$$
$$= \frac{1}{2}$$

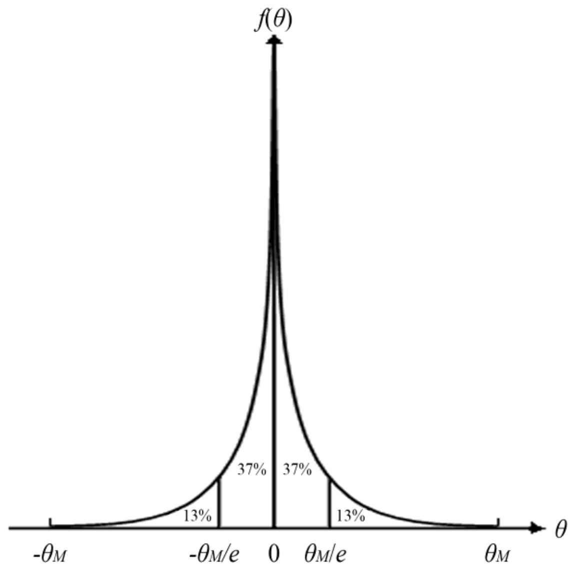

Fig.1 L probability density function

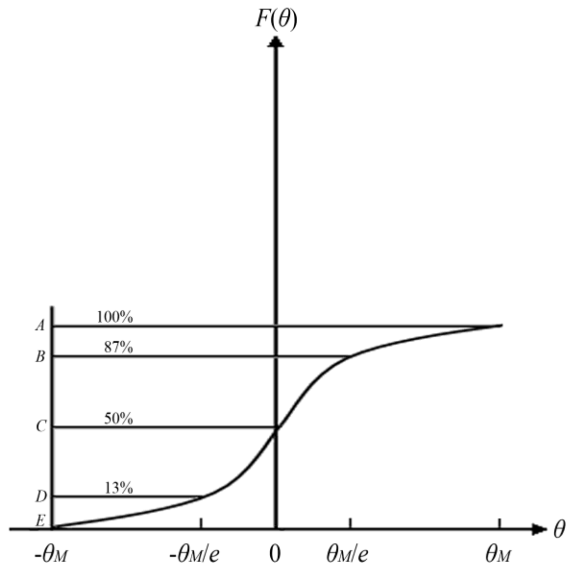

Fig.2 L probability distribution function

Above similarly through L'Hopital's Rule

$$\lim_{\theta\to 0}\ln(\theta)^{\frac{\theta}{2\theta_M}}=0$$

1.6 Demonstration of density function and distribution function

In fig 1 there

$P(-\theta_M/e < \theta < \theta_M/e) \approx 74\%$

$P(\theta_M/e < \theta < \theta_M) \approx 13\%$

$P(-\theta_M < \theta < \theta_M/e) \approx 13\%$

$(\theta_M-\theta_M/e)/\theta_M \approx 62.9\% \approx 61.8\%$(golden section)

$f(\theta)\to\infty, (\theta=0)$, $f(\theta)\to 0, (\theta=\pm\theta_M)$

In fig 2 also

$AB = DE = (1/2)-(1/e) = 13\%$

$BC = CD = (1/e) = 37\%$

$BD = BC+CD = (2/e) = 74\%$

$AE = AB+BC+CD+DE = 1-(2/e)+(2/e) = 100\%$

2. L distribution characteristics

2.1 Some important characteristics

Expected value

$$M(\theta)=\frac{1}{4\theta_M}\int_{-\theta_M}^{\theta_M}\theta\ln(\frac{\theta_M}{\theta})^2\,\mathrm{d}\theta=0\quad(-\theta_M<\theta<\theta_M)$$

Variance

$$\nu_2=D(\theta)=\frac{1}{4\theta_M}\int_{-\theta_M}^{\theta_M}\theta^2\ln(\frac{\theta_M}{\theta})^2\,\mathrm{d}\theta=(\frac{\theta_M}{3})^2$$

Standard deviation

$$\sigma=\frac{\theta_M}{3}$$

Third moment

$$\nu_3=\frac{1}{4\theta_M}\int_{-\theta_M}^{\theta_M}\theta^3\ln(\frac{\theta_M}{\theta})^2\,\mathrm{d}\theta=0$$

Coefficient of skew: $\frac{\nu_3}{\sigma^3} = 0$

Fourth moment: $\nu_4 = \frac{1}{4\theta_M}\int_{-\theta_M}^{\theta_M} \theta^4 \ln(\frac{\theta_M}{\theta})^2 \,d\theta = \frac{\theta_M^4}{25}$

Coefficient of kurtosis: $\frac{\nu_4}{\sigma^4} - 3 = 0.24$

2.2 Comparison between L distribution and Normal distribution

Probability of L distribution and normal distribution when θ in the context of σ; 2σ; 3σ, respectively, here σ denote respective standard deviation

L distribution

$$P(-\sigma < \theta < \sigma) = 0.70; \quad P(-2\sigma < \theta < 2\sigma) = 0.94; \quad P(-3\sigma < \theta < 3\sigma) = 1$$

Normal distribution [1~2]

$$P(-\sigma < x < \sigma) = 0.683; \quad P(-2\sigma < \theta < 2\sigma) = 0.955; \quad P(-3\sigma < \theta < 3\sigma) = 0.997$$

2.3 Comparison between L distribution and some other well-known distribution

Tab1 difference of some basic parameter for some distribution

The classification of distribution function	Mean	variance	4th central moment	CFK*
(Normal)	μ	σ^2	3.0	0
(Exponential)	$1/\lambda$	$1/\lambda^2$	9.0	6
(Gamma)	a/λ^2	a/λ^2	3+6a	b/a
**(Uniform)	$(a+b)/2$	$(a-b)^2/12$	1.8	-1.2
(Logarithm)	0	$(\theta_M/3)^2$	3.24	0.24

*CFK is abbreviated of Coefficient of Kurtosis. The curve of distribution becomes shape of sunken or hollow and depressed as CFK<-1.2.

2.4 Determination numerical value of $\theta = \theta_a(2/e) = \theta_a(74.04\%)$ and Where is θ as probability =2/e

Define follow

$$\int_{-\theta_M}^{\theta_a} f(\theta)d\theta = \frac{2}{e} + \frac{e-2}{2e} = \frac{1}{2} + \frac{1}{e};$$

$$\int_{-\theta_M}^{-\theta_a} f(\theta)d\theta = \frac{1}{2} - \frac{1}{e}$$

Also
$$\frac{1}{4\theta_M}\{[-\theta_a \ln(\frac{\theta_M}{\theta_a})^2 - 2\theta_a] + 2\theta_M\} = \frac{1}{2} - \frac{1}{e}$$
$$\frac{1}{4\theta_M}\{[\theta_a \ln(\frac{-\theta_M}{\theta_a})^2 + 2\theta_a] + 2\theta_M\} = \frac{1}{2} + \frac{1}{e}$$

Determine $\boldsymbol{\theta_a}$, finally by solution, θa (2/*e*)=±θ_M/e. Coefficient of skew is zero, imply L symmetric around zero, the coefficient of kurtosis 0.24, standard deviation $\boldsymbol{\theta_M/3}$, variance $\boldsymbol{(\theta_M/3)^2}$.

3. Summaries

In a word, L distribution with only one parameter $\boldsymbol{\theta_M}$ is shown as its mean 0 ,its variance $\boldsymbol{(\theta_M/3)^2}$, its standard deviation $\boldsymbol{\theta_M/3}$, its fourth moment $\boldsymbol{(\theta_M)^4/25}$, third moment zero, the coefficient of kurtosis 0.24, coefficient of skew is zero ,*m*-th moment $\boldsymbol{(\theta_M)^m/(m+1)^2}$, in addition, other characteristic, ratio of

$\boldsymbol{(\theta_M-\theta_M/e)/\ \theta_M}$ nearly to golden ratio, as well as $\boldsymbol{P(-\theta_M/e<\theta<\theta_M/e)=2/e\approx 74.04\%}$, similarly, $\boldsymbol{P(-\theta_M/3<\theta<\theta_M/3)\approx 70.0\%}$, in summary , continuous random variables of L function concentrically scatter in area near to its mean value than normal function does.

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