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Direct product of doubt intuitionistic fuzzy H-ideals in BCK/BCI-algebras

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Abstract

In this paper, we have enhanced the notion of the direct product of two intuitionistic fuzzy sets to the notion of the generalised direct product of two doubt intuitionistic fuzzy subalgebras and two doubt intuitionistic fuzzy H-ideals of two BCK/BCI-algebras using max-min operations. And we study some interesting properties of such direct product of doubt intuitionistic fuzzy H-ideals of BCK/BCI-algebras. Using level subsets of BCK/BCI-algebras, some characterization theorems are also given.

Keywords: BCK / BCI-algebra, Direct product, Doubt intuitionistic fuzzy subalgebra, Doubt intuitionistic fuzzy H-ideal.

1. Introduction

The concept of uncertainty has gone through a paradigmatic change in the last few decades and now it is not only an unavoidable plague to science and mathematics but it has, in fact, a great utility. In the evolution of this modified concept of uncertainty, the seminal work by L. A. Zadeh [12], had played a very crucial role. In his paper, Zadeh introduced a theory whose objects -fuzzy sets - are sets with boundaries that are not precise. Since the inception of this theory of fuzzy sets in mid - 1960s, its ramifications have been growing steadily. Extending this concept of fuzzy sets many researchers in later time, worked on various notions of higher order fuzzy sets. Atanossov's intuitionistic fuzzy sets [1, 2], is one among them. Many concepts in fuzzy set theory were extended in intuitionistic fuzzy set theory such as intuitionistic fuzzy relations, doubt intuitionistic fuzzy sets, intuitionistic L-fuzzy sets, intuitionistic fuzzy implications etc.

As a generalization of the concept of set-theoretic difference and proportional calculi, Imai and Iseki [7, 8, 9] introduced two classes of abstract algebra: BCI-algebras and BCK-algebras. It is known that the class of BCK-algebra is a proper subclass of the class of BCI-algebra. Since then, a great deal of literature has been produced on the theory of BCK/BCI-algebras. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups in 1971 by Rosenfeld [11] and later these ideas have been applied to other algebraic structures such as semigroups, rings, ideals, modules and vector spaces. Afterwards many researchers had worked on the structures of fuzzy sets in BCK/BCI-algebras and in other algebraic structures.

In [5], the authors have studied doubt intuitionistic fuzzy subalgebras and doubt intuitionistic fuzzy ideals in BCK/BCI-algebras, and also in [4, 6] the authors have studied doubt intuitionistic fuzzy H-ideals in BCK/BCI-algebras and doubt intuitionistic fuzzy sub-implicative ideals in BCI-algebras.

In 2001, Jun [10], introduced the direct product and T-product of T-fuzzy subalgebras. Thus there is a number of works on BCK/BCI-algebras and related algebraic systems but to the best of our knowledge no work is available on direct product of doubt intuitionistic fuzzy H-ideals in BCK/BCI-algebras. For this reason we are motivated to develop these theories for BCK/BCI-algebras.

Earlier we introduced the concept of doubt intuitionistic fuzzy subalgebras [5] and doubt intuitionistic fuzzy H-ideals [4] in BCK/BCI-algebras. In this paper, we define the direct product of two doubt intuitionistic fuzzy subalgebras and two doubt intuitionistic fuzzy H-ideals of two BCK/BCI-algebras. And also investigate some of its important properties. The noble relationship between them is also investigated. We have also found that the direct product of two intuitionistic fuzzy sets becomes doubt intuitionistic fuzzy H- ideals and doubt intuitionistic fuzzy subalgebras if and only if for any s, t $\in$ [0, 1], upper and lower level sets are H-ideals or subalgebras of BCK/BCI-algebra $X \times Y$.

2. Preliminaries

In this section, some elementary aspects that are necessary for this paper are included.

An algebra $(X; *, 0)$ of type (2, 0) is called a BCI-algebra if it satisfies the following axioms for all $x, y, z \in X$:

(A1) $((x * y) * (x * z)) * (z * y) = 0$

(A2) $(x * (x * y)) * y = 0$

(A3) $x * x = 0$

(A4) $x * y = 0$ *and* $y * x = 0$ imply $x = y$

If a BCI-algebra X satisfies $0 * x = 0$. Then X is called a BCK-algebra.

In a BCK/BCI-algebra, $x * 0 = x$ hold. A partial ordering "$\leq$" on a BCK/BCI-algebra X can be defined by $x \leq y$ if and only if $x * y = 0$.

Any BCK/BCI-algebra X satisfies the following axioms for all $x, y, z \in X$:

(i) $(x * y) * z = (x * z) * y$

(ii) $x * y \leq x$

(iii) $(x * y) * z \leq (x * z) * (y * z)$

(iv) $x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x$.

Throughout this paper, $X \times Y$ always means BCK/BCI-algebra without any specification.

A non-empty subset I of a BCK/BCI-algebra X is called an ideal of X if

(i) $0 \in I$

(ii) $x * y \in I$ and $y \in I$ then $x \in I$, for all $x, y \in X$.

A non-empty subset I of a BCK/BCI-algebra X is said to be a H-ideal of X if

(i) $0 \in I$

(ii) $x * (y * z) \in I$ and $y \in I$ then $x * z \in I$, for all $x, y, z \in X$.

A fuzzy set $A = \{(x, \alpha_A(x)) : x \in X\}$ in X is called a fuzzy H-ideal of X if

(i) $\alpha_A(0) \geq \alpha_A(x)$

(ii) $\alpha_A(x * z) \geq \min\{\alpha_A(x * (y * z)), \alpha_A(y)\}$, for all $x, y, z \in X$.

The propose work is done on intuitionistic fuzzy set. The formal definition of intuitionistic fuzzy set is given below:

An intuitionistic fuzzy set (briefly, IFS) A in a non-empty set X is an object having the form

$A = \{x, \alpha_A(x), \beta_A(x) / x \in X\}$, where the function $\alpha_A : X \to [0, 1]$ and $\beta_A : X \to [0, 1]$, denoted the degree of membership and the degree of non-membership of each element $x \in X$ to the set A respectively and $0 \leq \alpha_A(x) + \beta_A(x) \leq 1$, for all $x \in X$.

For the sake of simplicity, we use the symbol form $A = (X, \alpha_A, \beta_A)$ or (α_A, β_A) for the

intuitionistic fuzzy set $A = \{(x, \alpha_A(x), \beta_A(x)) : x \in X\}$.

The two operators used in this paper are defined as follows:

If A = (α_A, β_A) is an intuitionistic fuzzy set then,

$\Pi A = \{(x, \alpha_A(x), \overline{\alpha_A}(x))/x \in X\}$, where , $\overline{\alpha_A}(x) = 1 - \alpha_A(x)$.

$\blacklozenge A = \{(x, \overline{\beta_A}(x), \beta_A(x))/x \in X\}$, where, $\overline{\beta_A}(x) = 1 - \beta_A(x)$.

For the sake of simplicity, we also use $x \vee y$ for $\max(x, y)$, and $x \wedge y$ for $\min(x, y)$.

A fuzzy set A = $\{(x, \alpha_A(x)) : x \in X\}$ in X is called a doubt fuzzy subalgebra of X if

$\alpha_A(x * y) \leq \alpha_A(x) \vee \alpha_A(y)$, for all $x, y \in X$.

A fuzzy set A = $\{(x, \alpha_A(x)) : x \in X\}$ in X is called a doubt fuzzy ideal of X if

(i) $\alpha_A(0) \leq \alpha_A(x)$,

(ii) $\alpha_A(x) \leq \alpha_A(x * y) \vee \alpha_A(y)$, for all $x, y \in X$.

Let A = (α_A, β_A) be an intuitionistic fuzzy subset of a BCK/BCI-algebra X, then A is called a doubt intuitionistic fuzzy subalgebra(shortly DIF-subalgebra) of X if

(i) $\alpha_A(x * y) \leq \alpha_A(x) \vee \alpha_A(y)$,

(ii) $\beta_A(x * y) \geq \beta_A(x) \wedge \beta_A(y)$, for all $x, y \in X$.

Let A = (α_A, β_A) be an intuitionistic fuzzy subset of a BCK/BCI-algebra X, then A is called a doubt intuitionistic fuzzy H-ideal(shortly DIFH-ideal) [4] of X if

(i) $\alpha_A(0) \leq \alpha_A(x)$, $\beta_A(0) \geq \beta_A(x)$

(ii) $\alpha_A(x * z) \leq \alpha_A(x * (y * z)) \vee \alpha_A(y)$

(iii) $\beta_A(x * z) \geq \beta_A(x * (y * z)) \wedge \beta_A(y)$, for all $x, y, z \in X$.

Let A = (α_A, β_A) and B = (α_B, β_B) be two intuitionistic fuzzy sets in BCK/BCI-algebras X and Y respectively. Then direct product of IFSs [3] A and B is denoted by A×B = $(\alpha_{A\times B}, \beta_{A\times B})$ and defined as $\alpha_{A\times B}(x, y) = \min\{\alpha_A(x), \alpha_B(y)\}$ and $\beta_{A\times B}(x, y) = \max\{\beta_A(x), \beta_B(y)\}$ for all $(x, y) \in X \times Y$. That is the minimum function is used for the degree of membership and the maximum for the degree of non-membership.

Let X, Y be two BCK/BCI-algebras, then their Cartesian product $X \times Y = \{(x, y)/x \in X, y \in Y\}$ is also a BCK/BCI-algebra under the binary operation " $\star$ " defined in X × Y by $(x, y) \star (p, q) = (x \star p, y \star q)$, for all $(x, y), (p, q) \in X \times Y$.

3. Direct product of doubt intuitionistic fuzzy H-ideals in BCK/BCI-algebras

In this section, we consider the direct product of doubt intuitionistic fuzzy subalgebras and doubt intuitionistic fuzzy H-ideals of BCK/BCI-algebras. Before we study the product of doubt intuitionistic fuzzy subalgebras and doubt intuitionistic fuzzy H-ideals of BCK/BCI-algebras, we first define the product of doubt intuitionistic fuzzy subsets of X × Y.

Definition 3.1 Let X, Y be two BCK/BCI-algebras. Again let A = (α_A, β_A) and B = (α_B, β_B) be two doubt intuitionistic fuzzy sets in X and Y respectively. Then the direct product of DIFSs A and B is denoted by A × B = $(\alpha_{A\times B}, \beta_{A\times B})$, where $\alpha_{A\times B} : X \times Y \rightarrow [0, 1]$ is given by $\alpha_{A\times B}(x, y) = \max\{\alpha_A(x), \alpha_B(y)\}$ and $\beta_{A\times B}: X \times Y \rightarrow [0, 1]$ is given by $\beta_{A\times B}(x, y) = \min\{\beta_A(x), \beta_B(y)\}$ for all $(x, y) \in X \times Y$.

Definition 3.2 An intuitionistic fuzzy set A×B = $(\alpha_{A\times B}, \beta_{A\times B})$ of BCK/BCI-algebra X×Y is called a doubt intuitionistic fuzzy subalgebra of X × Y if

(A1) $\alpha_{A\times B}((x_1, y_1) * (x_2, y_2)) \leq \max\{\alpha_{A\times B}(x_1, y_1), \alpha_{A\times B}(x_2, y_2)\}$

(A2) $\beta_{A\times B}((x_1, y_1) * (x_2, y_2)) \geq \min\{\beta_{A\times B}(x_1, y_1), \beta_{A\times B}(x_2, y_2)\}$, for all $(x_1, y_1), (x_2, y_2) \in X \times Y$.

Theorem 3.3 Let A = (α_A, β_A) and B = (α_B, β_B) be two doubt intuitionistic fuzzy subalgebras of BCK/BCI-algebras X and Y respectively. Then A × B = $(\alpha_{A\times B}, \beta_{A\times B})$ is also a doubt intuitionistic fuzzy subalgebra of BCK/BCI-algebra X × Y .

Proof: For any $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then

$$\alpha_{A\times B}((x_1, y_1) * (x_2, y_2)) = \alpha_{A\times B}(x_1 * x_2, y_1 * y_2)$$
$$= \max\{\alpha_A(x_1 * x_2), \alpha_B(y_1 * y_2)\}$$
$$\leq \max\{\max\{\alpha_A(x_1), \alpha_A(x_2)\}, \max\{\alpha_B(y_1), \alpha_B(y_2)\}\}$$
$$= \max\{\max\{\alpha_A(x_1), \alpha_B(y_1)\}, \max\{\alpha_A(x_2), \alpha_B(y_2)\}\}$$
$$= \max\{\alpha_{A\times B}(x_1, y_1), \alpha_{A\times B}(x_2, y_2)\}$$

Again,

$$\beta_{A\times B}((x_1, y_1) * (x_2, y_2)) = \beta_{A\times B}(x_1 * x_2, y_1 * y_2)$$
$$= \min\{\beta_A(x_1 * x_2), \beta_B((y_1 * y_2)\}$$
$$\geq \min\{\min\{\beta_A(x_1), \beta_A(x_2)\}, \max\{\beta_B(y_1), \beta_B(y_2)\}\}$$
$$= \min\{\min\{\beta_A(x_1), \beta_B(y_1)\}, \max\{\beta_A(x_2), \beta_B(y_2)\}\}$$
$$= \min\{\beta_{A\times B}(x_1, y_1), \beta_{A\times B}(x_2, y_2)\}$$

Therefore, for all $(x_1, y_1), (x_2, y_2) \in X \times Y$, $A \times B$ is a doubt intuitionistic fuzzy subalgebra of BCK/BCI-algebra $X \times Y$. This completes the proof.

Theorem 3.4 Let $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ be two doubt intuitionistic fuzzy subalgebras of BCK/BCI-algebras X and Y respectively. Then

(i) $\alpha_{A\times B}(0, 0) \leq \alpha_{A\times B}(x, y)$ and

(ii) $\beta_{A\times B}(0, 0) \geq \beta_{A\times B}(x, y)$, for all $(x, y) \in X \times Y$.

Proof: By definition,

$\alpha_{A\times B}(0,0) = \alpha_{A\times B}\{(x,y) * (x,y)\} \leq \alpha_{A\times B}(x,y) \vee \alpha_{A\times B}(x,y) \leq \alpha_{A\times B}(x,y)$.

Therefore, $\alpha_{A\times B}(0,0) \leq \alpha_{A\times B}(x,y)$, for all $(x,y) \in X \times Y$.

Again,

$$\beta_{A\times B}(0,0) = \beta_{A\times B}\{(x,y) * (x,y)\} \geq \beta_{A\times B}(x,y) \wedge \beta_{A\times B}(x,y) \geq \beta_{A\times B}(x,y).$$

Therefore, $\beta_{A\times B}(0,0) \geq \beta_{A\times B}(x,y)$, for all $(x,y) \in X \times Y$.

Lemma 3.5 Let $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ be two doubt intuitionistic fuzzy subalgebras of BCK/BCI-algebras X and Y respectively. Then the following are true.

(i) $\alpha_A(0) \leq \alpha_B(y)$ and $\alpha_B(0) \leq \alpha_A(x)$, for all $x \in X, y \in Y$.

(ii) $\beta_A(0) \geq \beta_B(y)$ and $\beta_B(0) \geq \beta_A(x)$, for all $x \in X, y \in Y$.

Proof: Assume that $\alpha_B(y) < \alpha_A(0)$ and $\alpha_A(x) < \alpha_B(0)$, for $some\ x \in X\ and\ y \in Y$.

Then,

$$\alpha_{A\times B}(x, y) = \max\{\alpha_A(x), \alpha_B(y)\}$$
$$\leq \max\{\alpha_B(0), \alpha_A(0)\}$$
$$= \alpha_{A\times B}(0, 0)$$

That is a contradiction.

Similarly, let $\beta_A(x) > \beta_B(0)$ and $\beta_B(y) > \beta_A(0)$, for some for some $x \in X$ and $y \in Y$. Then,

$\beta_{A\times B}$ (x, y) = min[β_A (x), β_B (y)]

$\geq$ min[$\beta_B(0)$, $\beta_A(0)$]

= $\beta_{A\times B}$ (0, 0)

That is a contradiction. Thus proving the result.

Theorem 3.6 If A×B is a DIF-subalgebra of X×Y , then either A or B is a DIF-subalgebra of X × Y .

Proof: Since A × B is a DIF-subalgebra of X × Y , then for all $(x_1, y_1), (x_2, y_2) \in$ X × Y, we have

$\alpha_{A\times B}((x_1, y_1) * (x_2, y_2)) \leq \max\{\alpha_{A\times B}(x_1, y_1), \alpha_{A\times B}(x_2, y_2)\}$.

By putting $x_1 = x_2 = 0$, we get,

$\alpha_{A\times B}((0, y_1) * (0, y_2)) \leq \max\{\alpha_{A\times B}(0, y_1), \alpha_{A\times B}(0, y_2)\}$ (i)

Also, $\alpha_{A\times B}((0, y_1) * (0, y_2)) = \alpha_{A\times B}((0 * 0), (y_1 * y_2))$

$= max\{\alpha_A(0 * 0), \alpha_B(y_1 * y_2)\}$

$= \alpha_B(y_1 * y_2)$. (ii)

Again by using **Lemma3.5** we have, $\max\{\alpha_{A\times B}(0, y_1), \alpha_{A\times B}(0, y_2)\} = \max[\alpha_B(y_1), \alpha_B(y_2)]$ (iii)

So from (i), (ii) and(iii) we get, $\alpha_B(y_1 * y_2) \leq \max\{(y_1), \alpha_B(y_2)\}$.

Similar way we can prove, $\beta_B(y_1 * y_2) \geq \min\{\beta_B(y_1), \beta_B(y_2)\}$.

Hence B is a DIF-subalgebra of X × Y .

Definition 3.7 An intuitionistic fuzzy set A×B = $(\alpha_{A\times B}, \beta_{A\times B})$ of BCK/BCI-algebra X×Y is called a doubt intuitionistic fuzzy H-ideal of X × Y if

(A3) $\alpha_{A\times B}(0, 0) \leq \alpha_{A\times B}(x, y)$ and $\beta_{A\times B}(0, 0) \geq \beta_{A\times B}(x, y)$

(A4) $\alpha_{A\times B}((x_1, y_1) * (x_3, y_3)) \leq \max\{\alpha_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \alpha_{A\times B}(x_2, y_2)\}$

(A5) $\beta_{A\times B}((x_1, y_1) * (x_3, y_3)) \geq \min\{\beta_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \beta_{A\times B}(x_2, y_2)\}$, for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$.

Now, we investigate several properties of the newly defined direct products.

Theorem 3.8 Let A = (α_A, β_A) and B = (α_B, β_B) be two doubt intuitionistic fuzzy H-ideals of BCK/BCI-algebras X and Y respectively. Then A×B = $(\alpha_{A\times B}, \beta_{A\times B})$ is a doubt intuitionistic fuzzy H-ideal of BCK/BCI-algebra X × Y .

Proof: For any $(x, y) \in X \times Y$.

$\alpha_{A\times B}(0, 0) = \max\{\alpha_A(0), \alpha_B(0)\} \leq \max\{\alpha_A(x), \alpha_B(y)\} = \alpha_{A\times B}(x, y)$.

And, $\beta_{A\times B}(0, 0) = \min\{\beta_A(0), \beta_B(0)\} \geq \min\{\beta_A(x), \beta_B(y)\} = \beta_{A\times B}(x, y)$.

Now for any $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$,

$\alpha_{A\times B}((x_1, y_1) * (x_3, y_3)) = \alpha_{A\times B}(x_1 * x_3, y_1 * y_3)$

$= \max\{\alpha_A(x_1 * x_3), \alpha_B(y_1 * y_3)\}$

$\leq \max\{\max\{\alpha_A(x_1 * (x_2 * x_3)), \alpha_A(x_2)\}, \max\{\alpha_B(y_1 * (y_2 * y_3)), \alpha_B(y_2)\}\}$

$= \max\{\max\{\alpha_A(x_1 * (x_2 * x_3)), \alpha_B(y_1 * (y_2 * y_3))\}, \max\{\alpha_A(x_2), \alpha_B(y_2)\}\}$

$= \max\{\alpha_{A\times B}\{(x_1 * (x_2 * x_3)), (y_1 * (y_2 * y_3))\}, \alpha_{A\times B}(x_2, y_2)\}$

$\leq \max\{\alpha_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \alpha_{A\times B}(x_2, y_2)\}$

And

$\beta_{A\times B}((x_1, y_1) * (x_3, y_3)) = \beta_{A\times B}(x_1 * x_3, y_1 * y_3)$

$= \min\{\beta_A(x_1 * x_3), \beta_B(y_1 * y_3)\}$

$\geq \min\{\min\{\beta_A(x_1 * (x_2 * x_3)), \beta_A(x_2)\}, \min\{\beta_B(y_1 * (y_2 * y_3)), \beta_B(y_2)\}\}$

$= \min\{\min\{\beta_A(x_1 * (x_2 * x_3)), \beta_B(y_1 * (y_2 * y_3))\}, \min\{\beta_A(x_2), \beta_B(y_2)\}\}$

$= \min\{\beta_{A\times B}\{(x_1 * (x_2 * x_3)), (y_1 * (y_2 * y_3))\}, \beta_{A\times B}(x_2, y_2)\}$

$\geq \min\{\beta_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \beta_{A\times B}(x_2, y_2)\}$

Hence, $A \times B = (\alpha_{A\times B}, \beta_{A\times B})$ is a doubt intuitionistic fuzzy H-ideal of BCK/BCI-algebra $X \times Y$ for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$.

The above Theorem is verified by the following example.

Example 3.9 Let X = {0, 1, 2, 3} be a BCK-algebra with the following Cayley table:

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Let $A = (\alpha_A, \beta_A)$ be a doubt intuitionistic fuzzy H-ideal of X as defined by

X	0	1	2	3
α_A	0	0.3	0.2	0.3
β_A	1	0.7	0.8	0.7

Again, let $B = (\alpha_B, \beta_B)$ be a doubt intuitionistic fuzzy H-ideal of X as defined by

X	0	1	2	3
α_B	0.2	0.4	0.5	0.5
β_B	0.8	0.6	0.5	0.5

Obviously, X × X is also a BCK-algebra.

Here we get, $\alpha_{A\times B}(0, 0) = \alpha_{A\times B}(2, 0) = 0.2$. Also, $\alpha_{A\times B}(0, 1) = \alpha_{A\times B}(2, 1) = \alpha_{A\times B}(1, 1) = \alpha_{A\times B}(3, 1) = 0.4$. Again, $\alpha_{A\times B}(0, 2) = \alpha_{A\times B}(0, 3) = \alpha_{A\times B}(2, 2) = \alpha_{A\times B}(2, 3) = \alpha_{A\times B}(1, 2) = \alpha_{A\times B}(1, 3) = \alpha_{A\times B}(3, 3) = \alpha_{A\times B}(3, 2) = 0.5$. And, $\alpha_{A\times B}(1, 0) = \alpha_{A\times B}(3, 0) = 0.3$.

Also, $\beta_{A\times B}(0, 0) = \beta_{A\times B}(2, 0) = 0.8$. Also, $\beta_{A\times B}(0, 1) = \beta_{A\times B}(2, 1) = \beta_{A\times B}(1, 1) = \beta_{A\times B}(3, 1) = 0.6$. Again, $\beta_{A\times B}(0, 2) = \beta_{A\times B}(0, 3) = \beta_{A\times B}(2, 2) = \beta_{A\times B}(2, 3) = \beta_{A\times B}(1, 2) = \beta_{A\times B}(1, 3) = \beta_{A\times B}(3, 3) = \beta_{A\times B}(3, 2) = 0.5$. And, $\beta_{A\times B}(1, 0) = \beta_{A\times B}(3, 0) = 0.7$.

Then it is clear that A × B is a doubt intuitionistic fuzzy H-ideal of X × X.

Theorem 3.10 Let $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ be two doubt intuitionistic fuzzy H-ideals of BCK/BCI-algebras X and Y respectively. If A × B is a DIFH-ideal of X × Y, then A × B must be a doubt intuitionistic fuzzy subalgebra of X × Y.

Proof: Since A × B is a DIFH-ideal of X × Y , then for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$, we have

$\alpha_{A\times B}((x_1, y_1) * (x_3, y_3)) \leq \max\{\alpha_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \alpha_{A\times B}(x_2, y_2)\}$.

By putting $x_3 = y_3 = 0$, we get,

$$\alpha_{A\times B}(x_1, y_1) \leq \max\{\alpha_{A\times B}((x_1, y_1) * (x_2, y_2)), \alpha_{A\times B}(x_2, y_2)\} \quad \text{(i)}$$

Again since, $((x_1, y_1) * (x_2, y_2)) \leq (x_1, y_1)$, for all $(x_1, y_1), (x_2, y_2) \in X \times Y$.

$$\text{Then, } \alpha_{A\times B}((x_1, y_1) * (x_2, y_2)) \leq \alpha_{A\times B}(x_1, y_1) \quad \text{(ii)}$$

Hence from (i) and (ii) we get, $\alpha_{A\times B}((x_1, y_1) * (x_2, y_2)) \leq \alpha_{A\times B}(x_1, y_1) \leq \max\{\alpha_{A\times B}((x_1, y_1) * (x_2, y_2)), \alpha_{A\times B}(x_2, y_2)\} \leq \max\{\alpha_{A\times B}(x_1, y_1), \alpha_{A\times B}(x_2, y_2)\}$, for all $(x_1, y_1), (x_2, y_2) \in X \times Y$.

Similarly we can prove that, $\beta_{A\times B}((x_1, y_1) * (x_2, y_2)) \geq \min\{\beta_{A\times B}(x_1, y_1), \beta_{A\times B}(x_2, y_2)\}$, for all $(x_1, y_1), (x_2, y_2) \in X \times Y$. Thus A × B is a DIF-subalgebra of X × Y .

But the converse of **Theorem 3.10** may not be true.

Lemma 3.11 Let $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ be two doubt intuitionistic fuzzy H-ideals of BCK/BCI-algebras X and Y respectively. If A × B is a DIFH-ideal of X × Y , then the following are true.

(i) $\alpha_A(0) \leq \alpha_B(y)$ and $\alpha_B(0) \leq \alpha_A(x)$, for all $x \in X, y \in Y$.

(ii) $\beta_A(0) \geq \beta_B(y)$ and $\beta_B(0) \geq \beta_A(x)$, for all $x \in X, y \in Y$.

Proof: Proof is same as **Lemma 3.5**.

Lemma 3.12 If $A \times B = (\alpha_{A\times B}, \beta_{A\times B})$ is a doubt intuitionistic fuzzy H-ideal of BCK/BCI-algebra X×Y. If $(a, b) \leq (x, y)$, then $\alpha_{A\times B}(x, y) \leq \alpha_{A\times B}(a, b)$ and $\beta_{A\times B}(x, y) \geq \beta_{A\times B}(a, b)$, for all $(a, b), (x, y) \in X \times Y$.

Proof: Let $(a, b), (x, y) \in X \times Y$, such that $(a, b) \leq (x, y)$ implies $(a, b) * (x, y) = (0, 0)$. Now,

$$\begin{aligned}\alpha_{A\times B}(x, y) &= \alpha_{A\times B}((x, y) * (0, 0)) \\ &\leq \max\{\alpha_{A\times B}((x, y) * ((a, b) * (0, 0))), \alpha_{A\times B}(a, b)\} \\ &= \max\{\alpha_{A\times B}((x, y) * (a, b)), \alpha_{A\times B}(a, b)\} \\ &= \alpha_{A\times B}(a, b)\end{aligned}$$

And

$$\begin{aligned}\beta_{A\times B}(x, y) &= \beta_{A\times B}((x, y) * (0, 0)) \\ &\geq \min\{\beta_{A\times B}((x, y) * ((a, b) * (0, 0))), \beta_{A\times B}(a, b)\} \\ &= \min\{\beta_{A\times B}((x, y) * (a, b)), \beta_{A\times B}(a, b)\} \\ &= \beta_{A\times B}(a, b)\end{aligned}$$

This completes the proof.

Theorem 3.13 Let $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ be two DIFH-ideals of BCK/BCI-algebras X and Y respectively. Then $\Pi(A \times B) = (\alpha_{A\times B}, \bar{\alpha}_{A\times B})$ is a DIFH-ideal of X × Y , where, $\bar{\alpha}_{A\times B} = 1 - \alpha_{A\times B}$.

Proof: Since by **Theorem 3.8**, A × B is a DIFH-ideal of X × Y. Hence for any $(x, y) \in X \times Y$.

$\alpha_{A\times B}(0, 0) \leq \alpha_{A\times B}(x, y)$. Hence, $1 - \alpha_{A\times B}(0, 0) \geq 1 - \alpha_{A\times B}(x, y)$. That is $\bar{\alpha}_{A\times B}(0, 0) \geq \bar{\alpha}_{A\times B}(x, y)$.

Now for any $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$,

we have, $\alpha_{A\times B}((x_1, y_1) * (x_3, y_3)) \leq \max\{\alpha_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \alpha_{A\times B}(x2, y2)\}$.

Next, $1-\alpha_{A\times B}((x_1, y_1) * (x_3, y_3)) \geq 1-\max\{\alpha_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \alpha_{A\times B}(x_2, y_2)\}$.

That is, $\bar{\alpha}_{A\times B}((x_1, y_1) * (x_3, y_3)) \geq \min\{1-\alpha_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), 1-\alpha_{A\times B}(x_2, y_2)\}$.

Finally, $\bar{\alpha}_{A\times B}((x_1, y_1) * (x_3, y_3)) \geq \min\{\bar{\alpha}_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \bar{\alpha}_{A\times B}(x_2, y_2)\}$.

Hence, $\Pi(A \times B)$ is a DIFH-ideal of $X \times Y$.

Theorem 3.14 Let $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ be two DIFH-ideals of BCK/BCI-algebras X and Y respectively. Then $\blacklozenge(A \times B) = (\bar{\beta}_{A\times B}, \beta_{A\times B})$ is a DIFH-ideal of $X \times Y$, where, $\bar{\beta}_{A\times B} = 1 - \beta_{A\times B}$.

Proof: By **Theorem 3.8**, $A \times B$ is a DIFH-ideal of $X \times Y$. Hence for any $(x, y) \in X \times Y$.

$\beta_{A\times B}(0, 0) \geq \beta_{A\times B}(x, y)$. Hence, $1 - \beta_{A\times B}(0, 0) \leq 1 - \beta_{A\times B}(x, y)$. That is $\bar{\beta}_{A\times B}(0, 0) \leq \bar{\beta}_{A\times B}(x, y)$.

Now for any $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$, we have

$\beta_{A\times B}((x_1, y_1) * (x_3, y_3)) \geq \min\{\beta_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \beta_{A\times B}(x_2, y_2)\}$.

Next $1-\beta_{A\times B}((x_1, y_1) * (x_3, y_3)) \leq 1-\min\{\beta_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \beta_{A\times B}(x_2, y_2)\}$.

That is, $\bar{\beta}_{A\times B}((x_1, y_1) * (x_3, y_3)) \leq \max\{1-\beta_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), 1-\beta_{A\times B}(x_2, y_2)\}$. Finally, $\bar{\beta}_{A\times B}((x_1, y_1) * (x_3, y_3)) \leq \max\{\bar{\beta}_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \bar{\beta}_{A\times B}(x_2, y_2)\}$.

So, $\blacklozenge(A \times B)$ is a DIFH-ideal of $X \times Y$.

Theorem 3.15 Let $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ be two DIFH-ideals of BCK-algebras

X and Y respectively. Then $A \times B$ is a DIFH-ideals of BCK-algebras $X \times Y$ if and only if

$\Pi(A \times B) = (\alpha_{A\times B}, \bar{\alpha}_{A\times B})$ and $\blacklozenge(A \times B) = (\bar{\beta}_{A\times B}, \beta_{A\times B})$ are DIFH-ideals of $X \times Y$.

Proof: The proof follows from **Theorem 3.13** and **Theorem 3.14**.

Proposition 3.16 Let an intuitionistic fuzzy set $A \times B = (\alpha_{A\times B}, \beta_{A\times B})$ be a doubt intuitionistic fuzzy H-ideal of a BCK-algebra $X \times Y$. Then $\alpha_{A\times B}((0, 0) * ((0, 0) * (x, y))) \leq \alpha_{A\times B}(x, y)$ and

$\beta_{A\times B}((0,0) * ((0,0) * (x,y))) \geq \beta_{A\times B}(x,y)$, for all $(x,y) \in X \times Y$.

Proof:

$$\begin{aligned}
\alpha_{A\times B}((0, 0) * ((0,0) * (x, y))) &\leq \max\{\alpha_{A\times B}((0, 0) * ((x,y) * ((0,0) * (x,y)))), \alpha_{A\times B}(x, y)\} \\
&= \max\{\alpha_{A\times B}((0, 0) * ((x, y) * (0, 0)), \alpha_{A\times B}(x, y)\} \\
&= \max\{\alpha_{A\times B}((0,0) * (x,y)), \alpha_{A\times B}(x, y)\} \\
&= \max\{\alpha_{A\times B}((0, 0)), \alpha_{A\times B}(x, y)\} \\
&= \alpha_{A\times B}(x, y), \text{ for all } (x, y) \in X \times Y.
\end{aligned}$$

Therefore, $\alpha_{A\times B}((0,0) * ((0,0) * (x, y))) \leq \alpha_{A\times B}(x, y)$.

Again,

$$\begin{aligned}
\beta_{A\times B}((0,0) * ((0,0) * (x,y))) &\geq \min\{\beta_{A\times B}((0,0) * ((x,y) * ((0,0) * (x,y)))), \beta_{A\times B}(x, y)\} \\
&= \min\{\beta_{A\times B}((0,0) * ((x,y) * (0,0)), \beta_{A\times B}(x,y)\} \\
&= \min\{\beta_{A\times B}((0,0) * (x,y)), \beta_{A\times B}(x, y)\} \\
&= \min\{\beta_{A\times B}((0, 0)), \beta_{A\times B}(x, y)\}
\end{aligned}$$

$= \beta_{A\times B}$ (x, y), for all (x, y) $\in$ X $\times$ Y.

Therefore, $\beta_{A\times B}((0,0) * ((0,0) * (x,y))) \geq \beta_{A\times B}(x, y)$, for all (x, y) $\in$ X × Y .

This proposition does not hold for a BCI-algebra X × Y.

Corollary 3.17 Let A × B = ($\alpha_{A\times B}$, $\beta_{A\times B}$) be a doubt intuitionistic fuzzy H-ideal of a BCK/BCI-algebra X×Y . Then the sets, $D_{\alpha_{A\times B}} = \{(x, y) \in X\times Y / \alpha_{A\times B}(x, y) = \alpha_{A\times B}(0, 0)\}$, and

$D_{\beta_{A\times B}} = \{(x, y) \in X \times Y / \beta_{A\times B}(x, y) = \beta_{A\times B}(0, 0)\}$ are H-ideals of X.

Proof: Let A × B = ($\alpha_{A\times B}$, $\beta_{A\times B}$) be a doubt intuitionistic fuzzy H-ideal of X × Y . Obviously, (0, 0) $\in D_{\alpha_{A\times B}}$ and $D_{\beta_{A\times B}}$. Now, let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$, such that

$((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), (x_2, y_2) \in D_{\alpha_{A\times B}}$. Then $\alpha_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))) = \alpha_{A\times B}(0, 0) = \alpha_{A\times B}(x_2, y_2)$.

Now, $\alpha_{A\times B}((x_1, y_1) * (x_3, y_3)) \leq \max\{\alpha_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3)))\} = \alpha_{A\times B}(0, 0)$.

Again, since $\alpha_{A\times B}$is a doubt fuzzy H-ideal of X ×Y , $\alpha_{A\times B}(0, 0) \leq \alpha_{A\times B}((x_1, y_1) * (x_3, y_3))$.

Therefore, $\alpha_{A\times B}(0, 0) = \alpha_{A\times B}((x_1, y_1) * (x_3, y_3))$. It follows that, $((x_1, y_1) * (x_3, y_3)) \in D_{\alpha_{A\times B}}$, for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$. Therefore, $D_{\alpha_{A\times B}}$ is an H-ideal of X × Y. In the same way we can prove that $D_{\beta_{A\times B}}$is also an H-ideal of X × Y .

Theorem 3.18 If A × B is a DIFH-ideal of X × Y, then either A or B is a DIFH-ideal of

X × Y .

Proof: Since A × B is a DIFH-ideal of X × Y, then for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$, we have

$\alpha_{A\times B}((x_1, y_1) * (x_3, y_3)) \leq \max\{\alpha_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \alpha_{A\times B}(x_2, y_2)\}$.

By putting $y_1 = y_2 = y_3 = 0$, we get

$\alpha_{A\times B}((x_1, 0) * (x_3, 0)) \leq \max\{\alpha_{A\times B}((x_1, 0) * ((x_2, 0) * (x_3, 0))), \alpha_{A\times B}(x_2, 0)\}$ (i)

Also we have, $\alpha_{A\times B}((x_1, 0) * (x_3, 0)) = \alpha_{A\times B}((x_1 * x_3), (0 * 0)) = \max\{\alpha_A(x_1 * x_3), \alpha_B(0 * 0)\} =$

$\alpha_A(x_1 * x_3)$ (ii)

Similarly, $\alpha_{A\times B}((x_1, 0) * ((x_2, 0) * (x_3, 0))) = \alpha_A(x_1 * (x_2 * x_3))$ (iii)

Again by using **Lemma 3.11** we have, $\max\{\alpha_{A\times B}(x_1, 0), \alpha_{A\times B}(x_2, 0)\} = \max\{\alpha_A(x_1), \alpha_A(x_2)\}$ (iv)

$\alpha_{A\times B}(x_2, 0) = \alpha_A(x_2)$ (v)

So from (i), (ii), (iii), (iv) and (v) we get, $\alpha_A(x_1 * x_3) \leq \max\{\alpha_A(x_1 * (x_2 * x_3)), \alpha_A(x_2)\}$.

Similar way we can prove, $\beta_A(x_1 * x_3) \geq \min\{\beta_A(x_1 * (x_2 * x_3)), \beta_A(x_2)\}$. Hence A is a DIFH-ideal of X × Y.

4. Upper and lower level sets

Definition 4.1 Let A × B = ($\alpha_{A\times B}$, $\beta_{A\times B}$) be a doubt intuitionistic fuzzy H-ideal of a BCK/BCI-algebra X × Y ,

and s, t $\in$ [0, 1], Then α-level t-cut and β-level s-cut of A × B, is as follows:

$$\alpha^{\leq}_{A\times B,t} = \{(x, y) \in (X \times Y) / \alpha_{A\times B}(x, y) \leq t\}$$

$$\text{And } \beta^{\geq}_{A\times B,s} = \{(x, y) \in (X \times Y) / \beta_{A\times B}(x, y) \geq s\}.$$

Theorem 4.2 Let $A \times B = (\alpha_{A\times B}, \beta_{A\times B})$ be an intuitionistic fuzzy set of a BCK/BCI-algebra $X \times Y$, then $A\times B$ is a DIF-subalgebra of $X \times Y$ iff for any $s, t \in [0, 1]$, $\alpha^{\leq}_{A\times B,t}$ and $\beta^{\geq}_{A\times B,s}$ are subalgebras of $X \times Y$.

Proof: Assume that $A \times B$ be an intuitionistic fuzzy set of a BCK/BCI-algebra $X \times Y$. Now for any $s, t \in [0, 1]$ and $(x_1, y_1), (x_2, y_2) \in \alpha^{\leq}_{A\times B,t}$, we have $\alpha_{A\times B}(x_1, y_1) \leq t$ and also $\alpha_{A\times B}(x_2, y_2) \leq t$. Again let $A \times B$ is a DIF-subalgebra of $X \times Y$, then $\alpha_{A\times B}((x_1, y_1) * (x_2, y_2)) \leq \max\{\alpha_{A\times B}(x_1, y_1), \alpha_{A\times B}(x_2, y_2) \leq \max(t, t) = t$. Therefore it implies that $((x_1, y_1) * (x_2, y_2)) \in \alpha^{\leq}_{A\times B,t}$.

Similarly, for any $(x_1, y_1), (x_2, y_2) \in \beta^{\geq}_{A\times B,s}$, we have $\beta_{A\times B}(x_1, y_1) \geq s$ and also $\beta_{A\times B}(x_2, y_2) \geq s$, then $\beta_{A\times B}((x_1, y_1) * (x_2, y_2)) \geq \min\{\beta_{A\times B}(x_1, y_1), \beta_{A\times B}(x_2, y_2)\} \geq \min(s, s) = s$. Therefore it implies that $(x_1, y_1) * (x_2, y_2) \in \beta^{\geq}_{A\times B,s}$. Hence, $\alpha^{\leq}_{A\times B,t}$ and $\beta^{\geq}_{A\times B,s}$ are subalgebras of $X \times Y$.

Conversely, let $\alpha^{\leq}_{A\times B,t}$ and $\beta^{\geq}_{A\times B,s}$ are subalgebras of BCK/BCI-algebra $X \times Y$ and also let $A \times B$ is not a DIF-subalgebra of $X \times Y$. Then there exist $(x_1, y_1), (x_2, y_2) \in X \times Y$, such that $\alpha_{A\times B}((x_1, y_1) * (x_2, y_2)) > \max\{\alpha_{A\times B}(x_1, y_1), \alpha_{A\times B}(x_2, y_2)\}$. Now let $t_0 = \frac{1}{2}\{\alpha_{A\times B}((x_1, y_1) * (x_2, y_2)) + \max\{\alpha_{A\times B}(x_1, y_1), \alpha_{A\times B}(x_2, y_2)\}$.

This implies, $\alpha_{A\times B}((x_1, y_1) * (x_2, y_2)) > t_0 > \max\{\alpha_{A\times B}(x_1, y_1), \alpha_{A\times B}(x_2, y_2)\}$. So $(x_1, y_1) * (x_2, y_2) \notin \alpha^{\leq}_{A\times B,t}$. But $(x_1, y_1), (x_2, y_2) \in \alpha^{\leq}_{A\times B,t}$, . That is a contradiction. This completes the proof.

Theorem 4.3 If $\alpha^{\leq}_{A\times B,t}$ and $\beta^{\geq}_{A\times B,s}$ are either empty or H-ideals of $X \times Y$ for $t, s \in [0, 1]$. Then $A \times B$ is a DIFH-ideal of $X \times Y$.

Proof: Let $\alpha^{\leq}_{A\times B,t}$ and $\beta^{\geq}_{A\times B,s}$ be either empty or H-ideals of $X \times Y$ for $t, s \in [0, 1]$. For any $(x, y) \in X \times Y$, let $\alpha_{A\times B}(x, y) = t$ and $\beta_{A\times B}(x, y) = s$.

Then $(x, y) \in \alpha^{\leq}_{A\times B,t} * \beta^{\geq}_{A\times B,s}$, so $\alpha^{\leq}_{A\times B,t} \neq \varphi \neq \beta^{\geq}_{A\times B,s}$. Since $\alpha^{\leq}_{A\times B,t}$ and $\beta^{\geq}_{A\times B,s}$ are H-ideals of $X \times Y$, therefore $(0, 0) \in \alpha^{\leq}_{A\times B,t} * \beta^{\geq}_{A\times B,s}$. Hence, $\alpha_{A\times B}(0, 0)) \leq t = \alpha_{A\times B}(x, y)$ and $\beta_{A\times B}(0, 0) \geq s = \beta_{A\times B}(x, y)$, where $(x, y) \in X \times Y$. If there exist $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$ such that

$\alpha_{A\times B}(x_1, y_1) * (x_3, y_3)) > \max\{\alpha_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \alpha_{A\times B}(x_2, y_2)\}$, then by taking, $t_0 = \frac{1}{2}(\alpha_{A\times B}((x_1, y_1) * (x_3, y_3)) + \max\{\alpha_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \alpha_{A\times B}(x_2, y_2)\})$, we have,

$\alpha_{A\times B}((x_1, y_1) * (x_3, y_3)) > t_0 > \max\{\alpha_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \alpha_{A\times B}(x_2, y_2)\}$.

Hence,

$(x_1, y_1) * (x_3, y_3) \notin \alpha^{\leq}_{A\times B,t0}$, but $((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))) \in \alpha^{\leq}_{A\times B,t0}$ and $(x_2, y_2)\ \alpha^{\leq}_{A\times B,t0}$. That is $\alpha^{\leq}_{A\times B,t0}$ is not an H-ideal of $X \times Y$, which is a contradiction. Therefore, $\alpha_{A\times B}((x_1, y_1) * (x_3, y_3)) \leq \alpha_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))) * \alpha_{A\times B}(x_2, y_2)$, for any $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$.

Finally, assume that there exist $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$, such that $\beta_{A\times B}((x_1, y_1) * (x_3, y_3)) < \min\{\beta_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \beta_{A\times B}(x_2, y_2)\}$.

Taking $s_0 = \frac{1}{2}(\beta_{A\times B}((x_1, y_1) * (x_3, y_3)) + \min\{\beta_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \beta_{A\times B}(x_2, y_2)\})$,then

$\min\{\beta_{A\times B}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \beta_{A\times B}(x_2, y_2)\} > s_0 > \beta_{A\times B}((x_1, y_1) * (x_3, y_3))$.

Therefore, $((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))) \in \beta^{\geq}_{A\times B,s0}$ and $(x_2, y_2)\ \beta^{\geq}_{A\times B,s0}$ but $((x_1, y_1) * (x_3, y_3)) \notin \beta^{\geq}_{A\times B,s0}$. Again a contradiction. This completes the proof.

5. Conclusions

In this paper, we have outstretched the notion of the direct product of two intuitionistic fuzzy sets to the notion of the generalized direct product of two DIF-subalgebras and two DIFH- ideals of two BCK/BCI-algebras X and Y. One can generalize the same for any n BCK-algebras. We show that if A and B are two DIFH - ideals of X and Y then the direct product of A and B is also a DIFH- ideal of $X \times Y$. But the reverse may not hold. We have also found that the direct product of two intuitionistic fuzzy sets becomes DIFH- ideals and doubt intuitionistic fuzzy subalgebras if and only if for any s, t ∈ [0, 1], upper and lower level sets are H-ideals or subalgebras of BCK/BCI-algebra X × Y.

We expect that all the results proved in this paper can be proved for other algebraic structures.

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