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The Two Phase Flow Motions and their P. D. F. Representations

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Abstract

The two statistical representations, the Eulerian-Eulerian (EE) and Lagrangian-Eulerian (LE) representations are obtained in the framework of the probability density function (p.d.f.) formalism for the multiphase flow motions. The consistency relationships among fundamental statistical quantities in the EE and LE representations are obtained. The fundamental quantities in these statistical representations show a relationship to each other under conditions of spatial homogeneity. The transport equations for the probability densities in each statistical representation are obtained. The governing equations for the mean mass, mean momentum and second moment of velocity with respect to these two representations are derived from these transport equations. Particularly, for the EE representation, the p.d.f. formalism is shown to naturally lead to the generally used ensemble averaged equations for two-phase flow motions. The Galilean-invariant combinations of unclosed terms in the governing equations which need to be modelled are defined. The correspondence between unclosed terms in each statistical representation is obtained. The Hybrid EE-LE computations can benefit from this correspondence, which serves in consistently for inter-transferring the information. This analysis also serves as a guiding frame work for direct numerical simulations of two-phase flows that was avoided to measure the unclosed terms in the governing equations in these two statistical representations. Also, the advantages and limitations of these statistical representations are defined.

Keywords: P. D. F., Two Phase, Motion.

1. Introduction

In classical field theory the Lagrangian approach of the flow field is the way of analyzing the fluid motion where the analyst follows an individual fluid particle as it moves through space and time. By plotting the position of an individual particle through time, we can analyze the path of the particle. Whereas in the Eulerian approach of the flow field is a way of analyzing the fluid motion that focuses on specific locations in the space through which the fluid flows as time passes. The Lagrangian and Eulerian specifications of the flow field are sometimes denoted as the Lagrangian and Eulerian frame of reference. However, in general both the Lagrangian and Eulerian specification of the flow field can be applied in any observer's frame of reference and in any coordinate system used within the chosen frame of reference. In the Eulerian specification of a field, it is represented as a function of position x and time t. For example, the flow velocity is represented by a function $v=v(x\ (t),\ t) = v(x,\ t)$. On the other hand, in the Lagrangian specification, individual fluid particles are followed through time. The fluid particles are labeled by some time independent vector field x_0. Here x_0 can be taken as the center of mass of the particle at some initial time t_0.

The statistical representations of two phase flows are generally classified as Eulerian–Eulerian (EE) or Lagrangian–Eulerian (LE) approaches depending on the reference frames underlying their formulation. The EE statistical representation means a statistical approach where both the continuous and dispersed phases are described in a common Eulerian reference frame as Eulerian random fields. The LE statistical representation is a statistical approach that represents the dispersed phase in a Lagrangian frame by a number density based on the location of dispersed phase elements (DPE) centers [1].

The EE and LE statistical representations are essentially the description of a two phase flow in two reference frames, it is natural to expect that these representations are related. A major challenge in describing the two phase flows, therefore, is to establish the precise relationship

between these two modelling approaches. Now, on taking the conditions under which the said relationship holds and the conditions under which they do not need to be clearly established. By establishing the suitable form of the relationship between the two statistical representations has been given some usable conclusions. In 1998, Subramaniam et al. [6] had given the concept that computations of some two phase applications such as fuel sprays can potentially benefit by using the EE modelling approach in the near-nozzle region and the LE approach in the dispersed spray region. This can be shown by figure-1 which shows a schematic illustration of EE-field, LE-field and a handover from an EE-representation to a LE-representation in the spray [1].

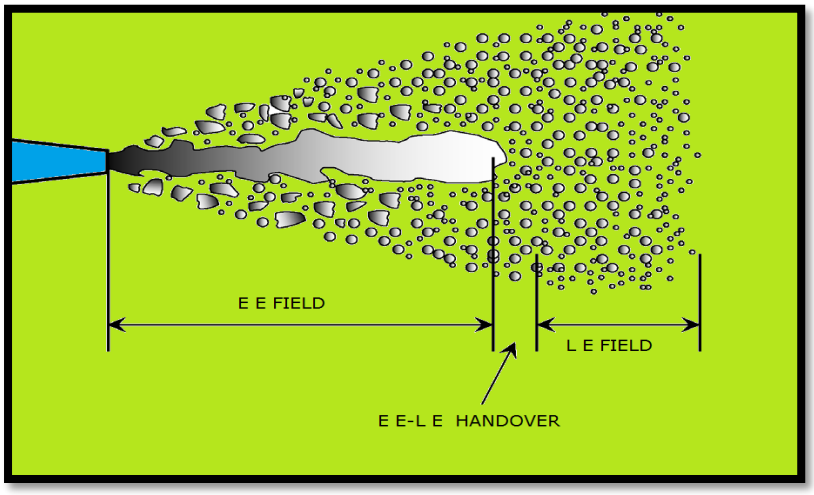

Figure-1.

(Schematic representations of a typical spray indicating the EE-region, LE-region and a handover between the EE and the LE descriptions. This handover requires consistency conditions to be satisfied between the two statistical representations at the common boundary of the two regions [1])

In the EE approach, the two phase flow field is represented as a random field while in the LE approach the dispersed phase is represented as a marked point process imbedded in a carrier flow. The related fundamental events and their corresponding probabilities associated with the two phase flow in the EE and LE framework have been discussed in this analysis. Now, the representations in both ways are as follows.

1.2. Random Field Representation

Let us consider a realization of a two phase flow with two distinct thermodynamic phases i.e. a carrier phase and a dispersed phase. Furthermore the term 'two phase flow' will be taken as an isothermal two phase flow without any reactions. Now each realization can be taken as an element of some sample space D that is the space of all possible real events. In a single realization and at a single space-time location, the phases are considered by using an indicator function $I_\alpha\,(x, t)$ for the α^{th} phase and will be defined as:

$I_\alpha\,(\boldsymbol{x}, t)$ =1, if $\boldsymbol{x}$ is in phase α at time t;

=0, if $\boldsymbol{x}$ is not in phase α at time t.

In two phase flows, the phase indicator functions satisfy the relation

$$\sum_{\alpha=\{c,d\}} I_\alpha\,(x,t) = 1. \qquad (1)$$

Where c represents the carrier phase and d represents the dispersed phase, for all the values (x, t). The instantaneous two phase velocity field $U(x, t)$, which is defined in both phases, is a vector field that is defined at each point x in the flow domain in physical space D. The schematic illustrations of the sample space with realizations are shown by the figure-2.

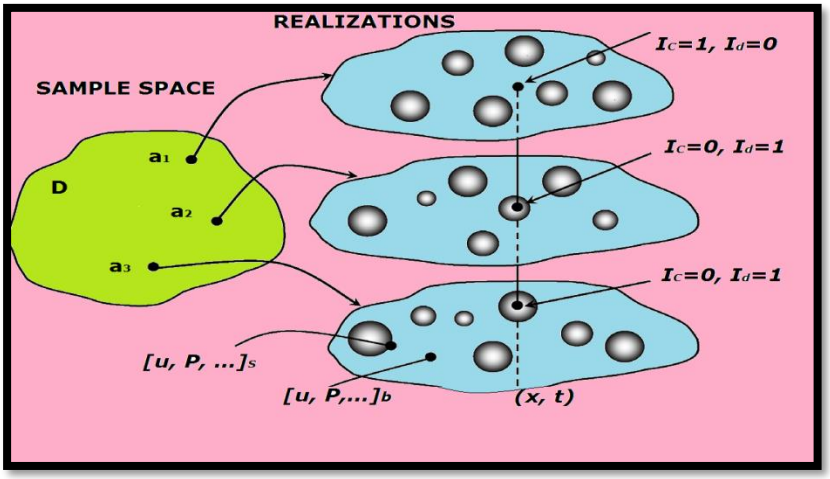

Figure-2.

(Schematic representation of the sample space D of all possible realizations of a two phase flow from which three realizations $\{a_1, a_2, a_3\}$ are given. The indicator function I_α (x, t) at a point (x, t), where

$\alpha = \{c, d\}$, as discussed in theory is shown for each of the three realizations. Also the primitive variables u for velocity and P for pressure at the DPE surface and in the bulk are shown by $[u, P, \ldots]_s$ and $[u, P, \ldots]_b$ respectively [1]).

The term $\rho(x, t)$ is the thermodynamic mass density function and is defined in both the phases. It has been considered that the density difference between the two phases is sufficiently large so that the density field can be used to distinguish between the two phases and the characteristic length scale of the interface over which this density change occurs is so small that in a continuum description the density changes discontinuously at the interface. But the phases are distinguished by the indicator function only then no information on shape or number of dispersed phase elements is available in this approach. Now the different events can be used to characterize the state of a two phase flow at a single space-time location (x, t), and each leads to different probabilities and p.d.f.'s. A complete Eulerian single-point p.d.f. description of the two phase flow will need the knowledge of the event i.e.

$$E_1 = [U \in (u, u + \delta u), \text{if } (x, t) = 1] \quad (2)$$

Which is the event corresponding to the joint occurrence of U in the range $(u, u + \delta u)$ at any point x and the fluid phase will present at the same point. Here u is the sample space variable corresponding to the random variable U. It will be noted that $I_c(x, t) = 1$ automatically precludes the occurrence of the dispersed phase at that same point i.e., $I_d(x, t) = 0$ at the same point x. Now for the joint event E_1, the two marginal events are

$$E_2 = [U(x, t) \in (u, u + \delta u)] \quad (3)$$

$$E_3(\alpha) = [I_\alpha(x, t) = 1] \quad (4)$$

Where E_2 is the event that $U(x, t)$ belongs to $(u, u + \delta u)$ without concern for whether the phase α is located at x, while $E_3(\alpha)$ is the event that phase α exists at x. The two conditional events are also useful and important is given by

$$E_4 = [U(x, t) \in (u, u + \delta u) | I_\alpha = 1] \quad (5)$$

$$E_5 = [I_\alpha(x, t) = 1 | U = u] \quad (6)$$

Where E_4 is the event that $U(x, t)$ belongs to $(u, u + \delta u)$ conditional on the presence of phase α at a point x, while E_5 is the event that the location x is occupied by the phase α with respect to condition on $U = u$ at the same location. Let the Eulerian p.d.f. of U be denoted as $f_U(u; x, t)$, where x and t are parameter space variables. The probabilities corresponding to each of the above events are given by

$$P[E_2] = P[U(x, t) \in (u, u + \delta u)] = f_U(u; x, t)\, \delta u \quad (7)$$

$$P[E_5] = P[I_\alpha (x, t) = 1|U = u] = P_\alpha(x, t|u) \quad (8)$$

$$P[E_1] = P[I_\alpha (x, t) = 1|U = u]\, P[U(x, t) \in (u, u + \delta u)] = P_\alpha (x, t|u)\, f_U(u; x, t)\, \delta u \quad (9)$$

$$P[E_3(\beta)] = P[I_\alpha = 1|U = u]\, f_U(u)\, \delta u = P_\alpha\, f_U(u)\, \delta u = A_\alpha (x, t) \quad (10)$$

$$P[E_4] = P[U(x, t) \in (u, u + \delta u)|I_\alpha = 1] = P_\alpha\, f_U(u; x, t)\, A_\alpha(x, t)\, \delta u, \quad (11)$$

Here $P_\alpha (x, t|u)$ is a phase probability function. Also, $A_\alpha(x, t)$ is the volume fraction at (x, t). The probability $P[E_3(\alpha)]$ defines a probability field $A_\alpha(x, t)$ is given as:

$$A_\alpha (x, t) \equiv P[I_\alpha (x, t) = 1] \quad (12)$$

Here $A_\alpha (x, t)$ is not a probability density function in x. It is a probability mass function in I_α that takes values $\{0, 1\}$. But f_U is a p.d.f. then it has to satisfy the condition of normalization i.e.,

$$\int f_U(u;\, x, t)\ \delta u\ =\ 1. \quad (13)$$

Also, let the probability $P[E_4]$ be denoted by $f_U|I_\alpha\ \delta u$, so that the Eulerian p.d.f. of velocity conditioned on the presence of phase α at x, $f_U|I_\alpha$ is given as:

$$f_U|I_\alpha = \frac{P_\alpha f_U(u)}{A_\alpha(x, t)} \quad (14)$$

The phase probability function P_α and the p.d.f. of instantaneous two phase velocity f_U can be written in terms of the volume fraction field A_α and the phasic velocity p.d.f. $f_U|I_\alpha$ as follows:

$$P_f(x, t|u) = \frac{A_c\ (x, t) f_{U|I_c}}{A_c\ (x, t) f_{U|I_c} + A_d\ (x, t) f_{U|I_d}} \quad (15)$$

$$f_U(u;\, x, t) = A_c\ (x, t) f_{U|I_c} + A_d\ (x, t) f_{U|I_d} \quad (16)$$

Hence these equations are sufficient for a complete single point description of a two phase flow motion.

1.3. Point Process Representation

In 1958, Williams et al., had given LE description of a two-phase flow [7]. The spray equation, which is the evolution equation of the d.d.f., can be accurately derived by starting from the Lagrangian evolution equations of droplet position, velocity and radius [3], [4]. But the d.d.f. was initially started to describe a fuel spray in internal combustion engines and hence the name 'droplet' distributions function was generated. It can be used to describe any two phase flow where the dispersed phase can be modelled as a collection of discrete distinct elements. Let us consider the DPEs to be droplets, but the discussion is equally valid for other DPEs. Also let us consider a two phase flow in a finite flow domain D in physical space as a collection of droplets. It is assumed that one can associate a characteristic length scale with each droplet that is the radius in the case of spherical droplets. If the droplet is non-spherical, then we will use the radius of an equivalent sphere that has the same volume as the non spherical droplet. We could also use the volume of the droplet directly as a phase space variable. However, either choice does not inherently alter the derivation of the spray equation nor does it provide any further insight into the nature of the unclosed terms in the spray equation and the moment equations derived thereof. Thus, we retain the radius as the characteristic length scale for the size phase space [5].

At time t the total number of droplets $n(t)$ is a non-negative integer valued random variable that is finite with probability 1. The i^{th} DPE is characterized by its position vector $x_i(t)$ which is defined as the centre of mass of the droplet, its velocity vector $v_i(t)$ and its radius $r_i(t)$. The position, velocity and radius of a droplet are called the droplet properties and the droplet property vector associated with each droplet is a seven-dimensional random vector in this representation. Any other additional droplet properties may be included as required but they do not fundamentally alter the

formulation other than increasing the dimension of the space of droplet properties. The properties associated with the i^{th} droplet given by the following equations:

$$\frac{dx_i}{dt} = v_i, \frac{dv_i}{dt} = a_i, \frac{dr_i}{dt} = \delta r_i \quad (17)$$

Where a_i is the acceleration experienced by the droplet, and δr_i is the rate of radius change due to vaporization or due to other reasons. This initial physical description for the LE approach assumes that the velocity field inside the droplet is uniform, and hence the motion of the i^{th} droplet can be described by the motion of its centre of mass x_i. The ensemble of droplets is characterized in the seven-dimensional position velocity radius space (x, v, r) by its fine grained

density function γ, which is defined as

$$\gamma(x, v, r, t) = \sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} \delta(x - x_i)\ \delta(v - v_i)\ \delta(r - r_i) \quad (18)$$

Here (x_i,v_i,r_i) are the Lagrangian coordinates of the i^{th} droplet whereas (x, v, r) are the sample-space coordinates. The function γ represents the density of droplets in a seven-dimensional (x, v, r) space. The summation of the product of delta functions in the above equation represents a single realization of the two phase flow. So the above summation represents a realization of the two phase flow in which the i^{th} DPE whose centre of mass x_i is at location x in position phase space, whose centre of mass velocity v_i is at location v in velocity phase space and whose radius r_i is at location r in radius phase space. If the number of droplets in any region B^+ in (x, v, r^+) space (since droplets with only non-zero radius belong to the spray system, if we denote r^+ to be the positive r-axis ($r > 0$), then it is suitable to integrate over regions only in (x, v, r^+) space) is denoted by $n(B^+; t)$, it is obtained by integrating γ over the region B^+ such that

$$n(B^+; t) = [\int \gamma(x, v, r, t)\ dx\ dv\ dr]_{B^+} \quad (19)$$

But γ is composed of delta functions it is not a smooth function in (x, v, r) space. The statistical description of a spray in terms of γ contains far more information than that is necessary for the calculations. So in order to find the information concerning the average properties of the spray, it is advantageous to consider the whole total average of γ. This average of γ is denoted by (x, v, r, t) and it defines the d.d.f. as given

$$\gamma_a(x, v, r, t) \equiv \langle \gamma(x, v, r, t) \rangle = \langle \sum_{i=1}^{n} \delta(x - x_i)\ \delta(v - v_i)\ \delta(r - r_i) \rangle \quad (20)$$

The expectation $\langle . \rangle$ in the above represents a whole total average of possibly infinite realizations of the two phase flow. The details on the use of the delta function to represent a realization of a single phase flow and its whole total average can be found. It is important to note that the expectation operator cannot be brought inside the summation for a general spray; if done, and then the conclusions of such an operation needs to understand [3].

The expected number of droplets $n(B^+; t)$ in a region B^+ of (x, v, r^+) space is given by

$$n(B^+;\ t) = \int_{B^+} \gamma_a(x, v, r, t) dx\ dv\ dr \quad (21)$$

The expected number of spray droplets $\langle n(t) \rangle$ at time t over the entire space (x, v, r^+) is given by

$$\langle n(t) \rangle = \int_{(x,v,r^+)} \gamma_a(x, v, r, t) dx\ dv\ dr \quad (22)$$

If the droplet distribution function is integrated over only (v, r^+) space, the density of the expected number of spray droplets $n_e(x; t)$ can be finding as follows:

$$n_e(x;\ t) = \int_{(v,r^+)} \gamma_a(x, v, r, t)\ dv\ dr \quad (23)$$

The joint p.d.f. i.e., j.p.d.f. for a p.d.f. conditional on *x* of velocity and radius can be taken as:

$$\gamma_{vr}^{x}(v,r;x;t) = \frac{\gamma_a(x,v,r,t)}{n_e(x;\ t)} \tag{24}$$

The 3-dimensional graphical representation of a j.p.d.f. is shown by the figure as follows:

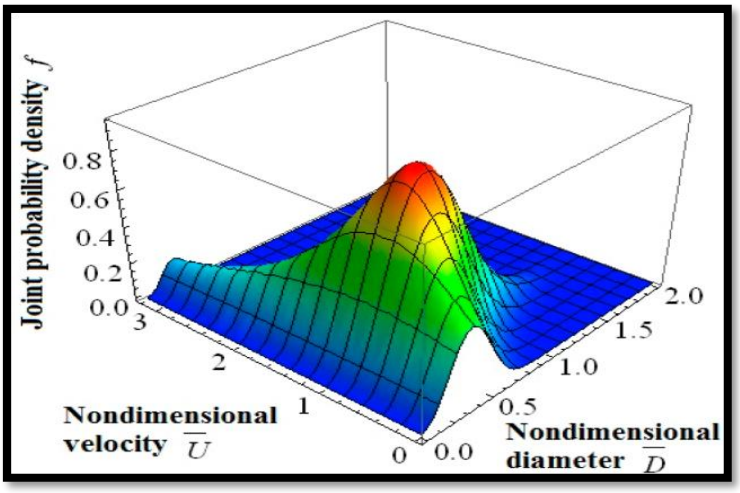

Figure-3.

Let us take,

$P[n(t)=k] = p_k$, γ^k denotes the density of expected number of droplets in the phase space with respect to the condition that the event $n(t)=k$ and $\gamma_{1s}^{k}(x, v, r; t)$ is the single-particle density of identically distributed considered droplets then we have

$$\gamma_a(x,v,r,t) = \sum_{k\geq 1} p_k \gamma^k\ (x,v,r;t) = \sum_{k\geq 1} k\ p_k \gamma_{1s}^{k}\ (x,v,r;t) \tag{25}$$

Hence we have the relation as given

$$\gamma_{vr}^{x}(v,r;x;t) = \frac{\sum_{k\geq 1} k\ p_k \gamma_{1s}^{k}\ (x,v,r;t)}{\sum_{k\geq 1}\ k\ p_k \gamma_{1s}^{k}\ (x;t)} \tag{26}$$

2. Graphical Representations of Some P.D.F.

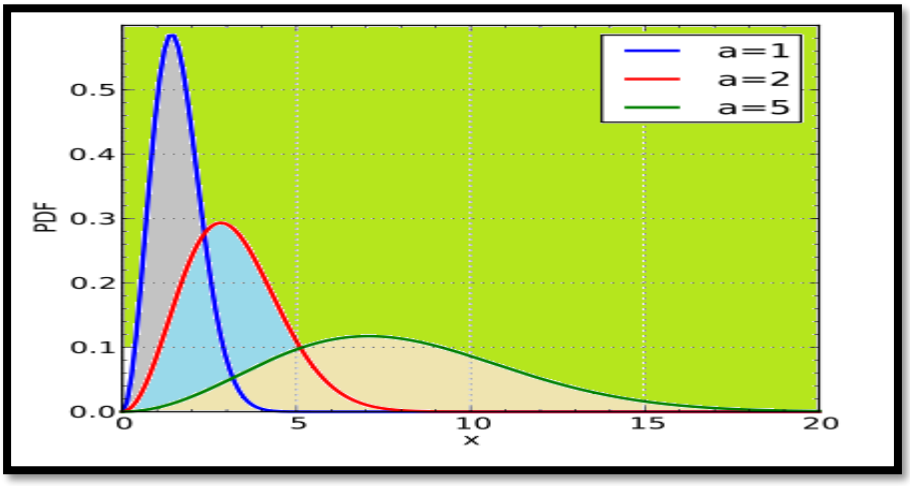

Figure-4.

(The graph between the variable *x* and *P.D.F. f(x)* for *a* =1, 2 and 5

$$P.D.F. = f(x) = \sqrt{\frac{2}{\pi}}\ \frac{x^2 e^{-x^2/2a^2}}{a^3})$$

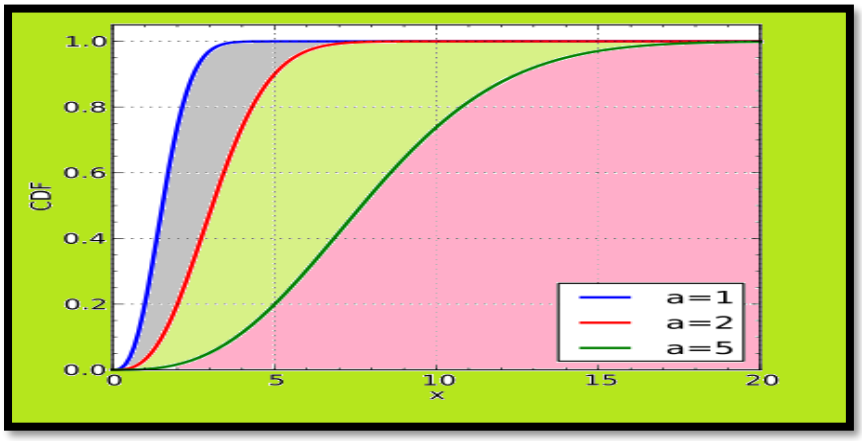

Figure-5.

(The graph between the variable *x* and Commulative *P.D.F. f(x)* for *a* =1, 2 and 5

$$C.P.D.F. = f(x) = \text{erf}\left(\frac{x}{a\sqrt{2}}\right) - \sqrt{\frac{2}{\pi}}\ \frac{x^2 e^{-x^2/2a^2}}{a})$$

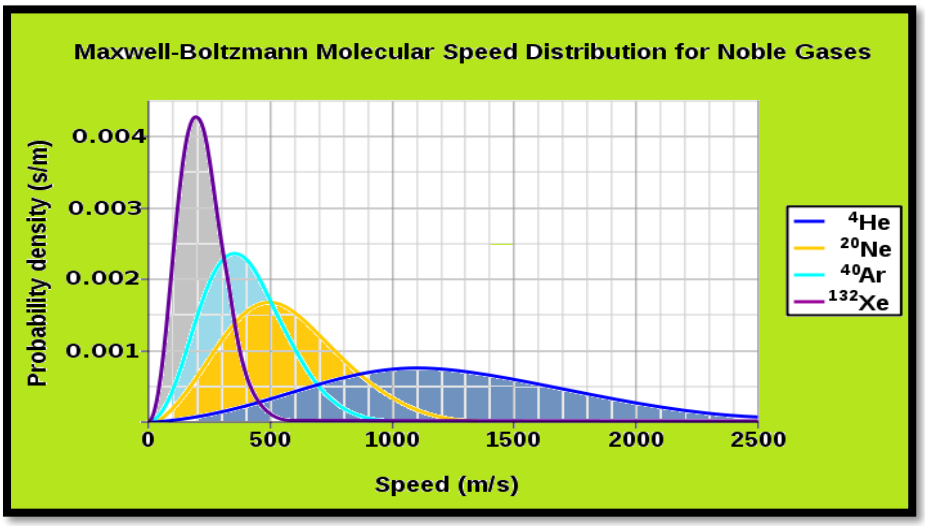

Figure-6.

(The speed probability density functions of the speeds of a few noble gases at a temperature of 298.15 K (25 °C). The *y*-axis is in s/m so that the area under any section of the curve that represents the probability of the speed being in that range is dimensionless)

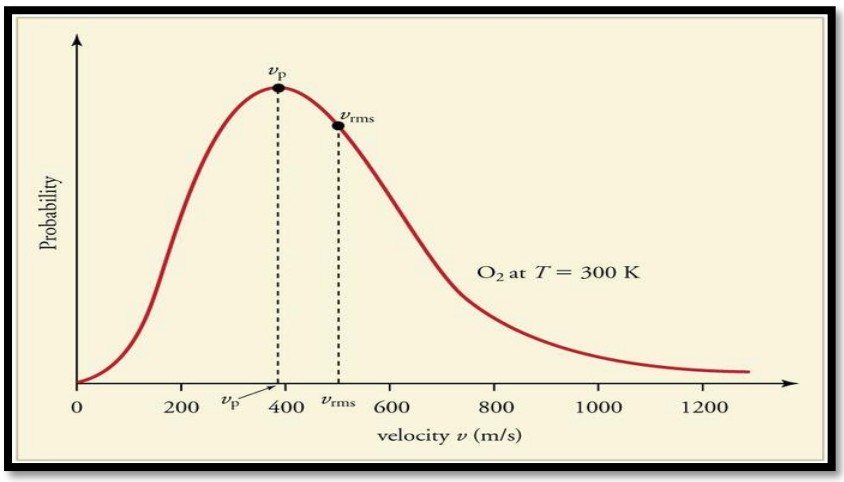

Figure-7.

(The P.D.F. of speeds for a certain gas at a certain temperature, such as nitrogen at 298 K. The speed at the top of the curve is called the most probable speed because the largest number of molecules has that speed) [2]

3. Conclusions

The two distinctly different statistical representations, namely the Eulerian–Eulerian and Lagrangian–Eulerian statistical representations, exist for a two-phase flow. It is clearly shown that the EE and LE probabilistic representations of two phase flow carry a complicated relationship with each other, which is different from simpler relationship between the Eulerian and Lagrangian descriptions in single phase flow. The fundamental events and corresponding probabilities associated with a two phase flow in the EE statistical representation have been founded. The governing equations for the mean mass, mean momentum and second moment that are derived from the given equation for the EE mass density are shown to be identical to widely used whole total averaged equations for two phase flows. The fundamental to the LE statistical representation is the droplet distribution function (d.d.f.) whose evolution equation has been accurately derived with the help of the theory of point processes. The transport equation forms the basis for the derivation of mean mass, mean momentum and second moment equations for the dispersed phase in the LE representation. The consistency conditions are established between the fundamental quantities in the EE and the LE statistical representations. By comparing unclosed terms in the governing equations for the mean mass, mean momentum and second moment in each statistical representation, correspondence between the unclosed terms is established. The comparison between the two statistical representations reveals that the information content in the two approaches is indeed different.

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