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Semi-simple Intuitionistic Fuzzy G - Modules

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Abstract

The concept of intuitionistic fuzzy G-modules and its properties including representation, reducibility and injectivity are already defined by the author et al. In this paper, we extend this idea to define semi-simplicity of intuitionistic fuzzy G-modules. The existence of a semi-simple intuitionistic fuzzy G-module for every finite dimensional G-module is proved and the relationships of semi-simplicity with other properties of intuitionistic fuzzy G-modules are also discussed.

Keywords: Intuitionistic fuzzy G-modules, Direct sum of intuitionistic fuzzy G-modules, Complete reducibility, Intuitionistic fuzzy injectivity, Semi-simple intuitionistic fuzzy G-modules.

AMS Classification: 03F55, 16D60.

1. Introduction

As a generalization of L.A. Zadeh's [13] fuzzy set, the concept of an intuitionistic fuzzy set was introduced by K.T. Atanassov [1], [2]. Applying this concept to algebra, R. Biswas [3] introduced the concept of intuitionistic fuzzy subgroups of a group and studied some of its properties. Later on many mathematicians worked on it and introduced the notion of intuitionistic fuzzy subrings, intuitionistic fuzzy modules etc. [6], [7] and [8]. Intuitionistic fuzzification of G-modules are made by the author et al. in [9]. Many properties like representation, complete reducibility and injectivity of intuitionistic fuzzy G-modules are discussed in [10], [11] and [12] respectively.

In this paper, we define semi-simplicity of intuitionistic fuzzy G-modules using direct sum of intuitionistic fuzzy G-modules. We prove the existence of semi-simple intuitionistic fuzzy G-modules on every finite dimensional G-module. We also obtain the relationship between complete reducibility and semi-simplicity of intuitionistic fuzzy G-modules and relate intuitionistic fuzzy injectivity with intuitionistic fuzzy semi-simplicity.

2. Preliminaries

Throughout this article, concepts and notation related with G-modules are mainly taken from [4], [5] and concepts and notation related with intuitionistic fuzzy G-modules are taken from [9], [10], [11] and [12].

Let G be a group and M be a vector space over a field K. Then M is called a **G-module** if for every $g \in G$ and $m \in M$, $\exists$ a product (called the action of G on M), $gm \in M$ satisfies the following axioms

i) $1_G \cdot m = m, \forall m \in M$ (1_G being the identity of G)

ii) $(g \cdot h) \cdot m = g \cdot (h \cdot m), \forall m \in M, g, h \in G$

iii) $g\cdot(k_1m_1 + k_2m_2) = k_1(g\cdot m_1) + k_2(g\cdot m_2)$, $\forall k_1, k_2\in$ K; $m_1, m_2\in$ M and $g\in$ G.

A subspace of M, which itself is a G-module with the same action is called **G-submodule** of M. It can be seen that the intersection of G-submodules is again a G-submodule. A non-zero G-module M is **irreducible** if the only G-submodules of M are M and {0}. Otherwise it is reducible. A non-zero G-module M is **completely reducible** if for every G-submodule N of M, there exists a G-submodule N* of M such that M = N ⊕ N*. It is well known that G-submodules of completely reducible G-modules are completely reducible. For G-modules M and M*, M is **M*- injective** if, for every submodule N* of M*, any homomorphism φ from N* to M can be extended as a homomorphism ψ from M* to M. A G-module M is **semi simple** if there exists a family of irreducible G-submodules M_i such that $M = \overset{n}{\underset{i=1}{\oplus}} M_i$. It is evident that completely reducible G-modules are semi simple.

Let G be a group and M be a G-module over K, which is a subfield of **C.** Then an **intuitionistic fuzzy G-module** on M is an intuitionistic fuzzy set A = (μ_A , ν_A) of M such that following conditions are satisfied

(i) $\mu_A(ax + by) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(ax + by) \leq \nu_A(x) \vee \nu_A(y)$, $\forall$ $a, b\in$ K and $x, y\in$ M and

(ii) $\mu_A(gm) \geq \mu_A(m)$ and $\nu_A(gm) \leq \nu_A(m)$, $\forall$ $g \in$ G ; $m \in$ M.

The standard intuitionistic fuzzy intersection of finite number of intuitionistic fuzzy G-modules is again an intuitionistic fuzzy G-module, while standard union and compliment need not be so. If $M = \overset{n}{\underset{i=1}{\oplus}} M_i$ is a G-module and A_i is an intuitionistic fuzzy G-module on M_i $\forall$ i, then $\overset{n}{\underset{i=1}{\oplus}} A_i$ is defined by $\left(\overset{n}{\underset{i=1}{\oplus}} A_i\right)(x) = \left(\mu_{\overset{n}{\underset{i=1}{\oplus}} A_i}(x),\ \nu_{\overset{n}{\underset{i=1}{\oplus}} A_i}(x)\right)$, where

$\mu_{\overset{n}{\underset{i=1}{\oplus}} A_i}(x) = \min\left\{\mu_{A_i}(x_i) : i = 1,\ 2, ..., n\right\}$ and $\nu_{\overset{n}{\underset{i=1}{\oplus}} A_i}(x) = \max\left\{\nu_{A_i}(x_i) : i = 1,\ 2, ..., n\right\}, \forall\ x = \sum_{i=1}^{n} x_i \in M, x_i \in M_i.$
is an intuitionistic fuzzy G-module on M called the **direct sum of intuitionistic fuzzy G-modules** A_i, i =1,2,…,n. An intuitionistic fuzzy G-module A on M is **completely reducible** if

(i) M is completely reducible,

(ii) M has at least one proper G-submodule and

(iii) Corresponding to any proper decomposition $M = M_1 \oplus M_2$, there exists intuitionistic fuzzy G-modules A_i on M_i, i = 1, 2, such that $A = A_1 \oplus A_2$ with $\wedge(A_1) \neq \wedge(A_2)$ [i.e., set of double pinned flags for the intuitionistic fuzzy G-module $A_1 \neq$ set of double pinned flags for the intuitionistic fuzzy G-module A_2].

Let M and M^* be G-modules. Let A = (μ_A, ν_A) be any intuitionistic fuzzy G-module on M and B = (μ_B, ν_B) be any intuitionistic fuzzy G-module on M^*. Then **A is B-injective** if

(i) M is M^*- injective.

(ii) $\mu_B(m) \leq \mu_A(\psi(m))$ and $\nu_B(m) \geq \nu_A(\psi(m))$, $\forall\ \psi \in$ Hom (M^*, M) and $m \in M^*$.

3. Semi–simple intuitionistic fuzzy G-modules

Definition (3.1) An intuitionistic fuzzy G-module A on M is said to be semi-simple if M is semi-simple G-module with decomposition $M = \overset{n}{\underset{i=1}{\oplus}} M_i$ and $A = \overset{n}{\underset{i=1}{\oplus}} A_i$, where A_i is an intuitionistic fuzzy G-module on M_i , $\forall$ i.

Example (3.2) Let G = {1, -1} and M = Q($\sqrt{2}$) over Q . Then M is semi-simple G-module with M = Q($\sqrt{2}$) = Q $\oplus$ $\sqrt{2}$ Q. Let A be an intuitionistic fuzzy set on M be defined by

$$\mu_A(a+\sqrt{2}\ b) = \begin{cases} 1 & if\ a=b=0 \\ 1/2 & if\ a \neq 0 \text{ and } b=0 \\ 1/5 & if\ b \neq 0 \end{cases} \quad \text{and} \quad \nu_A(a+\sqrt{2}\ b) = \begin{cases} 0 & if\ a=b=0 \\ 1/4 & if\ a \neq 0 \text{ and } b=0. \\ 1/2 & if\ b \neq 0 \end{cases}$$

Define intuitionistic fuzzy sets A_1 and A_2 on Q and $\sqrt{2}$ Q as follows:

$$\mu_{A_1}(a) = \begin{cases} 1 & if\ a=0 \\ 1/2 & if\ a \neq 0 \end{cases}, \quad \nu_{A_1}(x) = \begin{cases} 0 & if\ a=0 \\ 1/4 & if\ a \neq 0 \end{cases};\ \forall\ a \in Q \text{ and}$$

$$\mu_{A_2}(\sqrt{2}b) = \begin{cases} 1 & if\ b=0 \\ 1/5 & if\ b \neq 0 \end{cases}, \quad \nu_{A_2}(\sqrt{2}b) = \begin{cases} 0 & if\ b=0 \\ 1/2 & if\ b \neq 0 \end{cases};\ \forall\ b \in Q.$$

Then A_1 and A_2 are intuitionistic fuzzy G-modules on Q and $\sqrt{2}$ Q respectively such that $\mu_A(a+\sqrt{2}\ b) = \mu_{A_1}(a) \wedge \mu_{A_2}(\sqrt{2}b)$ and $\nu_A(a+\sqrt{2}\ b) = \nu_{A_1}(a) \vee \nu_{A_2}(\sqrt{2}b),\ \forall\ a+\sqrt{2}\ b \in M.$

Therefore, $A = A_1 \oplus A_2$. Hence A is a semi-simple intuitionistic fuzzy G-module on M.

Proposition (3.3) Let M be a semi-simple G-module with decomposition $M = \overset{n}{\underset{i=1}{\oplus}} M_i$. If $A_1 = \overset{n}{\underset{j=1}{\oplus}} A_{1_j}$ and $A_2 = \overset{n}{\underset{j=1}{\oplus}} A_{2_j}$ are two semi-simple intuitionistic fuzzy G-modules on M, then $A_1 \cap A_2$ is also a semi-simple intuitionistic fuzzy G-module on M, where $\cap$ denotes standard intuitionistic fuzzy intersection.

Proof: The standard intuitionistic fuzzy intersection of fuzzy G- modules is an intuitionistic fuzzy G-module defined by $(A_1 \cap A_2)(x) = \left(\mu_{A_1 \cap A_2}(x),\ \nu_{A_1 \cap A_2}(x)\right)$, where

$\mu_{A_1\cap A_2}(x)=\min\left\{\mu_{A_1}(x),\ \mu_{A_2}(x)\right\}$ and $\nu_{A_1\cap A_2}(x)=\max\left\{\nu_{A_1}(x),\ \nu_{A_2}(x)\right\},\ \forall\ x=\sum_{i=1}^{n}x_i\in M.$

$$\begin{aligned}
\mu_{A_1\cap A_2}(x)&=\min\left\{\mu_{A_1}(x),\ \mu_{A_2}(x)\right\}\\
&=\min\left\{\min\{\mu_{A_{11}}(x_1),\mu_{A_{12}}(x_2),\ \ldots\ldots,\ \mu_{A_{1n}}(x_n)\},\ \min\{\mu_{A_{21}}(x_1),\mu_{A_{22}}(x_2),\ \ldots\ldots,\ \mu_{A_{2n}}(x_n)\}\right\}\\
&=\min\left\{\min\{\mu_{A_{11}}(x_1),\mu_{A_{21}}(x_1)\},\min\{\mu_{A_{12}}(x_2),\mu_{A_{22}}(x_2)\},\ldots\ldots\ldots\ldots,\min\{\mu_{A_{1n}}(x_n),\mu_{A_{2n}}(x_n)\}\right\}\\
&=\min\left\{\mu_{A_{11}\cap A_{21}}(x_1),\mu_{A_{12}\cap A_{22}}(x_2),\ldots\ldots\ldots\ldots,\mu_{A_{1n}\cap A_{2n}}(x_n)\right\}\\
&=\min\left\{\mu_{B_1}(x_1),\mu_{B_2}(x_2),\ldots\ldots\ldots\ldots,\mu_{B_n}(x_n)\right\},\text{ where } \mathrm{B}_i=A_{1i}\cap A_{2i} \text{ is an IFG-module on } \mathrm{M}_i,\forall\ i.\\
&=\mu_{\overset{n}{\underset{i=1}{\oplus}}B_i}(x)
\end{aligned}$$

and

$$\begin{aligned}
\nu_{A_1\cap A_2}(x)&=\max\left\{\nu_{A_1}(x),\ \nu_{A_2}(x)\right\}\\
&=\max\left\{\max\{\nu_{A_{11}}(x_1),\nu_{A_{12}}(x_2),\ \ldots\ldots,\ \nu_{A_{1n}}(x_n)\},\ \max\{\nu_{A_{21}}(x_1),\nu_{A_{22}}(x_2),\ \ldots\ldots,\ \nu_{A_{2n}}(x_n)\}\right\}\\
&=\max\left\{\max\{\nu_{A_{11}}(x_1),\nu_{A_{21}}(x_1)\},\max\{\nu_{A_{12}}(x_2),\nu_{A_{22}}(x_2)\},\ldots\ldots\ldots\ldots,\max\{\nu_{A_{1n}}(x_n),\nu_{A_{2n}}(x_n)\}\right\}\\
&=\max\left\{\nu_{A_{11}\cap A_{21}}(x_1),\nu_{A_{12}\cap A_{22}}(x_2),\ldots\ldots\ldots\ldots,\nu_{A_{1n}\cap A_{2n}}(x_n)\right\}\\
&=\max\left\{\nu_{B_1}(x_1),\nu_{B_2}(x_2),\ldots\ldots\ldots\ldots,\nu_{B_n}(x_n)\right\},\text{ where } \mathrm{B}_i=A_{1i}\cap A_{2i} \text{ is an IFG-module on } \mathrm{M}_i,\forall\ i.\\
&=\nu_{\overset{n}{\underset{i=1}{\oplus}}B_i}(x).
\end{aligned}$$

So, $A_1\cap A_2=\overset{n}{\underset{i=1}{\oplus}}B_i$, where $B_i=A_{1i}\cap A_{2i}$ is an intuitionistic fuzzy G-module on $\mathrm{M}_i,\forall\ i=1,2,...,n.$
Hence $A_1\cap A_2$ is a semi-simple intuitionistic fuzzy G-module on M.

Proposition (3.4) Any finite dimensional G- module with dimension at least 2 has a semi-simple intuitionistic fuzzy G-module.

Proof: Assume that M is a G-module with dimension n ≥ 2, and $\{m_1, m_2,\ldots, m_n\}$ is a basis for M.

Let M_i = span $\{m_i\}$. Then M is semi-simple with $M=\overset{n}{\underset{i=1}{\oplus}}M_i$.

Define an intuitionistic fuzzy set A on M by

$$\mu_A(\mathrm{c}_1\mathrm{m}_1+\mathrm{c}_2\mathrm{m}_2+\ldots\ldots\ldots+\mathrm{c}_n\mathrm{m}_n)=\begin{cases}1 & ;\ if\ \mathrm{c}_i=0\ \forall\ i\\ 1/2 & ;\ if\ \mathrm{c}_1\neq 0,\ \mathrm{c}_2=\mathrm{c}_3=0,\ldots..,\mathrm{c}_n=0\\ 1/3 & ;\ if\ \mathrm{c}_2\neq 0,\ \mathrm{c}_3=\mathrm{c}_4=0,\ldots..,\mathrm{c}_n=0\\ \ldots. & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ 1/n & ;\ if\ \mathrm{c}_{n-1}\neq 0,\ \mathrm{c}_n=0\\ 1/n+1 & ;\ if\ \mathrm{c}_n\neq 0\end{cases}$$

and

$$\nu_A(c_1m_1+c_2m_2+..........+c_nm_n)=\begin{cases} 0 & ;\ if\ c_i=0\ \forall\ i \\ 1/n+1; & if\ c_1\neq 0,\ c_2=0,.....,c_n=0 \\ 1/n & ;\ if\ c_2\neq 0,\ c_3=0,.....,c_n=0 \\ .... & .... \\ 1/3 & ;\ if\ c_{n-1}\neq 0,\ c_n=0 \\ 1/2 & ;\ if\ c_n\neq 0 \end{cases}.$$

Define intuitionistic fuzzy sets A_i on M_i by

$$\mu_{A_i}(x)=\begin{cases} 1 & if\ x=0 \\ 1/i+1 & if\ x\neq 0 \end{cases} \text{ and } \nu_{A_i}(x)=\begin{cases} 0 & if\ x=0 \\ 1/n-i+2 & if\ x\neq 0 \end{cases} ;\ \forall\ x\in M_i\ .$$

Then it can be easily checked that

$\mu_A(m)=\min\{\mu_{A_i}(c_im_i):i=1,2,....,n\}$ and $\nu_A(m)=\max\{\nu_{A_i}(c_im_i):i=1,2,....,n\}$, where $m=\sum_{i=1}^{n}c_im_i\in$ M.

Thus $A=\oplus_{i=1}^{n}A_i$, hence the result.

4. Semi-simplicity and other properties

The semi-simplicity of an intuitionistic fuzzy G-module is related to properties like complete reducibility and intuitionistic fuzzy injectivity of intuitionistic fuzzy G-modules. These relationships are derived in the following propositions

Proposition (4.1) For any finite dimensional G-module M, semi-simple intuitionistic fuzzy G-modules on M are completely reducible.

Proof: Let A be a semi-simple intuitionistic fuzzy G-module on M. Assume that $M=\oplus_{i=1}^{n}M_i$ and $A=\oplus_{i=1}^{n}A_i$ where A_i are intuitionistic fuzzy G-modules on irreducible G-submodules M_i of M.
Let N be any G-submodule of M. Then N is spanned by the elements $\{m_1, m_2,…..........,m_s\}$ of a basis $\{m_1, m_2, …,m_s, m_{s+1}, …., m_n\}$ of M. Let N′ be the submodule spanned by the remaining basis vectors.
Then M = N ⊕ N′ and for any $x=\sum_{i=1}^{n}x_i\in M$, we have

$$\mu_A(x)=\min\left\{\mu_{A_1}(x_1),\mu_{A_2}(x_2),...........,\mu_{A_n}(x_n)\right\}$$
$$=\ \min\left\{\min\{\mu_{A_1}(x_1),\mu_{A_2}(x_2),\ ......,\ \mu_{A_s}(x_s)\},\ \min\{\mu_{A_{s+1}}(x_{s+1}),\mu_{A_{s+2}}(x_{s+2}),\ ......,\ \mu_{A_n}(x_n)\}\right\}$$
$$=\min\left\{\mu_{\oplus_{i=1}^{s}A_i}(x),\ \mu_{\oplus_{i=s+1}^{n}A_i}(x)\right\}$$
$$=\min\left\{\mu_{B_1}(x),\mu_{B_2}(x)\right\}, \text{where } B_1=\oplus_{i=1}^{s}A_i \text{ and } B_2=\oplus_{i=s+1}^{n}A_i \text{ are IFG-modules on N and N}'$$
$$=\mu_{B_1\oplus B_2}(x)$$

and

$$\nu_A(x) = \max\left\{\nu_{A_1}(x_1), \nu_{A_2}(x_2), \ldots\ldots\ldots\ldots, \nu_{A_n}(x_n)\right\}$$

$$= \max\left\{\max\{\nu_{A_1}(x_1), \nu_{A_2}(x_2), \ \ldots\ldots, \ \nu_{A_s}(x_s)\}, \ \max\{\nu_{A_{s+1}}(x_{s+1}), \nu_{A_{s+2}}(x_{s+2}), \ \ldots\ldots, \ \nu_{A_n}(x_n)\}\right\}$$

$$= \max\left\{\nu_{\oplus_{i=1}^{s} A_i}(x), \ \nu_{\oplus_{i=s+1}^{n} A_i}(x)\right\}$$

$$= \max\left\{\nu_{B_1}(x), \ \nu_{B_2}(x)\right\}, \text{where } B_1 = \mathop{\oplus}_{i=1}^{s} A_i \text{ and } B_2 = \mathop{\oplus}_{i=s+1}^{n} A_i \text{ are IFG-modules on N and N}'$$

$$= \nu_{B_1 \oplus B_2}(x).$$

Thus $A = B_1 \oplus B_2$. This shows that A is completely reducible.

Proposition (4.2) A completely reducible intuitionistic fuzzy G-module A on n dimensional G-module M is semi-simple if $\mu_A\left(\sum_{i=1}^{n} x_i\right) = \min\{\mu_A(x_1), \mu_A(x_2), \ldots\ldots\ldots\ldots\ldots\ldots, \mu_A(x_n)\}$ and $\nu_A\left(\sum_{i=1}^{n} x_i\right) = \max\{\nu_A(x_1), \nu_A(x_2), \ldots\ldots\ldots, \nu_A(x_n)\}$, for every $\sum_{i=1}^{n} x_i \in M$.

Proof. Since A is completely reducible, so M is completely reducible and hence M is semi-simple. Let M_i be the G-submodule of M spanned by the basis vector $\{m_i\}$ of a basis $\{m_1, m_2, \ldots, m_n\}$ of M.

Then $M = \mathop{\oplus}_{i=1}^{n} M_i$, and let $x = \sum_{i=1}^{n} x_i \in M$ be any element. Then

$$\left.\begin{aligned} \mu_A(x) &= \mu_A\left(\sum x_i\right) = \min\{\mu_A(x_1), \mu_A(x_2), \ldots\ldots\ldots\ldots\ldots\ldots, \mu_A(x_n)\} \text{ and} \\ \nu_A(x) &= \nu_A\left(\sum x_i\right) = \max\{\nu_A(x_1), \nu_A(x_2), \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \nu_A(x_n)\} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

As A is completely reducible, for the decomposition $M_1 \oplus N_1$ of M, where $N_1 = \mathop{\oplus}_{i=2}^{n} M_i$, A is decomposed into $A = A_1 \oplus A_1'$, where A_1 and A_1' are intuitionistic fuzzy G-modules on M_1 and N_1 respectively.

Hence $\mu_A(x) = \min\{\mu_{A_1}(x_1), \mu_{A_1'}(x_1')\}$ and $\nu_A(x) = \max\{\nu_{A_1}(x_1), \nu_{A_1'}(x_1')\}$, where $x_1' = x_2 + x_3 + \ldots + x_n$.

Similarly, for every decomposition $M_i \oplus N_i$ of M, we can find intuitionistic fuzzy G-modules A_I and A_I' so that

$\mu_A(x) = \min\{\mu_{A_i}(x_i), \mu_{A_i'}(x_i')\}$ and $\nu_A(x) = \max\{\nu_{A_i}(x_i), \nu_{A_i'}(x_i')\}$.....(2), where $x_i' \in N_i, i = 1, 2, \ldots, n$.

Each of these n equations gives the inequalities

$\mu_A(x) \le \min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots\ldots., \mu_{A_n}(x_n)\}$ and $\nu_A(x) \ge \max\{\nu_{A_1}(x_1), \nu_{A_2}(x_2), \ldots\ldots, \nu_{A_n}(x_n)\}$.....(3)

Equation (2) gives that $\mu_A(x_i) = \mu_{A_i}(x_i)$ and $\nu_A(x_i) = \nu_{A_i}(x_i)$ which together with (1) proves

$\mu_A(x) \ge \min\left\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots\ldots., \mu_{A_n}(x_n)\right\}$ and $\nu_A(x) \le \min\left\{\nu_{A_1}(x_1), \nu_{A_2}(x_2), \ldots\ldots., \nu_{A_n}(x_n)\right\}$..(4)

Equation (3) and (4) together gives

$\mu_A(x) = \min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_n}(x_n)\}$ and $\nu_A(x) = \max\{\nu_{A_1}(x_1), \nu_{A_2}(x_2), ..., \nu_{A_n}(x_n)\}$

Thereby making $A = \oplus_{i=1}^{n} A_i$, where A_i's are intuitionistic fuzzy G-modules on M_i. This proves that A is semi-simple.

Proposition (4.3) If M* is a semi-simple G-module, then M is M*-injective for every G-module M.

Proof. Semi-simplicity of M* gives $M^* = \oplus_{i=1}^{n} M_i$. Let N* be any G-submodule of M* and φ be a homomorphism from N* to M.

Case (i) If N* = {0}, then φ = 0 and ψ = 0 is an extension of φ from M* to M.

Case (ii) If N* = M_i, then $\psi(c_1m_1 + c_2m_2 + ... + c_nm_n) = \varphi(c_i m_i)$ is an extension of φ from M* to M.

Case (iii) If $N^* = \oplus_{i=1}^{k} M_i$, k < n, then

$\psi(c_1m_1 + c_2m_2 + ... + c_nm_n) = \varphi(c_1m_1 + c_2m_2 + ... + c_km_k)$ gives the required extension.

This proves that every G-submodule M is M*-injective.

Proposition (4.4) If G is a finite group and B is a semi-simple intuitionistic fuzzy G-module on M*, then for any intuitionistic fuzzy G-module A on M, A is B - injective if and only if A is B_i - injective for every i.

Proof. Since B is a semi-simple intuitionistic fuzzy G-module on M* . So, $M^* = \oplus_{i=1}^{n} M_i$, and $B = \oplus_{i=1}^{n} B_i$, where B_i is an intuitionistic fuzzy G-module on M_i.

Let us first assume that A is B-injective, then we have

(i) M is M*- injective and

(ii) $\mu_B(m) \leq \mu_A(\psi(m))$ and $\nu_B(m) \geq \nu_A(\psi(m))$, $\forall\ \psi \in$ Hom (M^*, M) and $m \in M^*$.

Since M is M*- injective and M_i is a G-submodule of M. Therefore, M is M_i-injective, $\forall\ i = 1, 2, ..., n$ and (iii) $\mu_B(m_i) = \mu_{B_i}(\psi(m_i))$ and $\nu_B(m_i) = \nu_{B_i}(\psi(m_i))$ $\forall\ i = 1, 2, ..., n.$

Let ψ be any homomorphism in Hom(M_i, M). As M is M*- injective, every homomorphism from M_i to M can be extended as a homomorphism from M* to M.

Let φ is an extension of ψ to Hom(M*, M). Then from (i), (ii) and (iii), we get

$\mu_{B_i}(m_i) \leq \mu_A(\varphi(m_i)) = \mu_A(\psi(m))$ and $\nu_{B_i}(m_i) \geq \nu_A(\varphi(m_i)) = \nu_A(\psi(m))$, $\forall\ \psi \in$ Hom(M*, M).

This proves that A is B_i-injective for every $i = 1,2,.....,n$.

Conversely, assume that A is B_i-injective for every $i = 1,2,.....,n$. Then by proposition (4.3) we have M is M*-injective. Let $\psi \in$ Hom (M^*, M) and $m \in M^*$. Then $m = \sum_{i=1}^{n} m_i$, $m_i \in M_i$.

Now, $\mu_B(m) = \min\left\{\mu_{B_1}(m_1), \mu_{B_2}(m_2), ..., \mu_{B_n}(m_n)\right\} \leq \mu_{B_i}(m_i)$

and $\nu_B(m) = \max\left\{\nu_{B_1}(m_1), \nu_{B_2}(m_2), ..., \nu_{B_n}(m_n)\right\} \geq \nu_{B_i}(m_i), \ \forall\ i = 1, 2, ..., n.$

Since A is B_i- injective, therefore we have

$\mu_{B_i}(m_i) \leq \mu_A(\psi(m_i))$ and $\nu_{B_i}(m_i) \geq \nu_A(\psi(m_i)), \ \forall\ i = 1, 2, ..., n.$

Hence $\mu_B(m) \leq \min\{\mu_A(\psi(m_i)) : i = 1, 2, ..., n\}$ and $\nu_B(m) \geq \max\{\nu_A(\psi(m_i)) : i = 1, 2, ..., n\}$.

Therefore, $\mu_{B_i}(m) \leq \mu_A\{\psi(m_1) + \psi(m_1) + ... + \psi(m_1)\} = \mu_A(\psi(m))$

and $\nu_{B_i}(m) \geq \nu_A\{\psi(m_1) + \psi(m_1) + ... + \psi(m_1)\} = \nu_A(\psi(m))$.

i.e., $\mu_{B_i}(m) \leq \mu_A(\psi(m))$ and $\nu_{B_i}(m) \geq \nu_A(\psi(m)), \ \forall\ \psi \in Hom(M^*, \mathrm{M})$ and $m \in M^*$.

Thus A is $\mathrm{B}_i - injective.$

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